THE UNIVERSITY OF ALBERTA

THE APPROXIMATE POWER OF A TEST FOR SERIAL INDEPENDENCE IN A STATIONARY LINEAR MARKOV PROCESS

BY

© JOHN HANSON

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA
OCTOBER, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Approximate Power of a Test for Serial Independence in a Stationary Linear Markov Process", submitted by John Hanson in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The purpose of this thesis is to obtain the approximate power of a test for serial independence in a stationary linear Markov process. The approximate power is obtained for a test based on a sample estimate of the serial correlation coefficient in a stationary linear Markov process with unknown mean. Chapter I introduces the problem and Chapter II reviews Patrick's derivation of the approximate distribution function of a sample serial correlation coefficient in a circular universe. Then, in Chapter III, following Patrick's approach, the approximate distribution function of a sample serial correlation coefficient for the stationary case is derived. This is used in Chapter IV to obtain the approximate power of the test for serial independence.

ACKNOWLEDGEMENTS

I would like to thank Dr. J. R. McGregor for his assistance in the preparation of this thesis.

TABLE OF CONTENTS

TABLE

TABLE OF FIGURES

CHAPTER I

INTRODUCTION

The recent interest in the problem of testing for serial correlation in linear Markov processes seems to have been initiated by a paper of R. L. ANDERSON (1942) . In this paper, using the circular definition of the serial correlation coefficient as introduced by Hotelling, the author derived a useful test of significance and gave a table of significance points for the lag - L serial correlation coefficient. A summary of some of the many papers that have since been published dealing with this topic can be found in ABBASSI (1964) .

The essential problem of significance testing is the selection of an effective testing procedure. As a criterion for comparing tests, NEYMAN and PEARSON (1933) introduced the notion of the power of a test. The power is the probability that the test will reject the tested hypothesis when it is false. Hence, when choosing a good test, we seek one with large power. With this in view, the object of this thesis is to derive an approximate expression for the distribution function of a sample serial correlation coefficient in a linear Markov process and thus obtain the approximate power of a test of serial independence.

Consider the linear Markov process in discrete time defined by the stochastic difference equation

$$x_s = \rho\, x_{s-1} + e_s$$

for all s , where the real constant ρ is called the "lag one serial correlation coefficient" and the $[\, e_s \,]$ are independently and identically distributed, $N(0\ ,\ 1)$, random variables. In the present thesis, we assume the linear Markov process to be stationary, that is, $|\rho| < 1$. Let $x_1 , \ldots , x_n$ be a sample of n observations from the process. As our sample estimate of the serial correlation coefficient ρ we take the ratio

$$(1.1) \qquad \hat{\rho} = \frac{C_o}{C}$$

where C_o and C are the following quadratic forms in the observations,

$$(1.2) \quad \begin{cases} C_o = \frac{1}{2} x_1^2 + x_1 x_2 + \ldots + x_{n-1} x_n + \frac{1}{2} x_n^2 - n\bar{x}^2 \\ C = x_1^2 + \ldots + x_n^2 - n\bar{x}^2 \end{cases}$$

and

$$\bar{x} = \frac{x_1 + \ldots + x_n}{n} .$$

PATTON (1961) derived the approximate distribution of the sample serial correlation coefficient defined above using the method of DANIELS (1956) . The approximate renormalized density function was found to be

$$h(\hat{\rho}) \sim \frac{K(n,\rho)(1 - \hat{\rho}^2)^{\frac{N}{2}}}{(1 - 2\rho\hat{\rho} + \rho^2)^{\frac{N-1}{2}}} \; \{ 1 + O (n^{-\frac{3}{2}}) \}$$

where

$$K(n,\rho) = \frac{2}{\sqrt{\pi}} \; \frac{\Gamma (\frac{N + 5}{2})}{\Gamma(\frac{N}{2} + 1) \; [(1 - \rho^2) \; (N - 1) + 4 \;]}$$

and

$$N = n - 1 + \frac{\rho^2}{1 - \rho^2} .$$

This result was used to obtain an approximate expression for the distribution of the ratio of the square successive difference to the square difference,

$$D = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2} .$$

The statistic D is related to $\hat{\rho}$ by $\hat{\rho} = 1 - \frac{1}{2} D$ and for large n is similar to the ratio of the mean square successive difference to the variance,

$$\eta = \frac{n \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{(n-1) \sum_{i=1}^{n} (x_i - \bar{x})^2} .$$

The distribution of the statistic η has been discussed by VON NEUMANN (1941) and (1942) and WILLIAMS (1941), for the case where the samples are taken to consist of independent normal random variables $x_1, \ldots, x_n$. Patton was able to obtain an approximate expression for the distribution of η by letting ρ equal zero in the approximation derived for the distribution of D.

In the present paper we take the approach of PATRICK (1964) to obtain an approximate expression for the distribution function of the sample serial correlation coefficient $\hat{\rho}$. This approach incorporates the methods of GURLAND (1948) and DANIELS (1956) with the asymptotic techniques developed by ERDELYI and WYMAN (1963) and (1963a) for deriving asymptotic expansions for complex integrals in terms of Confluent Hypergeometric functions. Briefly, to Gurland's method of deriving inversion formulae for characteristic functions is applied Daniels' method of steepest descent to obtain a complex integral representing the distribution function of the sample serial correlation coefficient. The techniques of Erdelyi and Wyman are then used to obtain an asymptotic expansion for the complex integral in terms of the Confluent Hypergeometric function.

Our review of Patrick's method omits his development for deriving the asymptotic expansion of the integral

$$\int_{-1\,[L^*]}^{+1} \frac{\chi(t)}{\rho - r - it\sqrt{1 - r^2}} (1 - t^2)^s \, dt .$$

In the present paper we simply assume his results, which may be found in detail, in PATRICK (1964), Appendix III.

Chapter II is a review of Patrick's approach for the case of a serial correlation coefficient defined in a circular universe, and in Chapter III we obtain an approximate expression for the distribution function of the sample serial correlation coefficient defined above. In Chapter IV, we find the approximate power of a test of serial independence, based on our sample estimate of the serial correlation coefficient, for different alternatives ρ and sample size n. These results are used to obtain the power curves for the various sample sizes.

CHAPTER II

THE APPROXIMATE DISTRIBTUION FUNCTION OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT IN A CIRCULAR UNIVERSE

This chapter serves as a review of the asymptotic approach adopted by PATRICK (1964) to obtain approximate expressions for distribution functions of sample estimates of serial correlation coefficients in linear Markov processes. For this purpose, we consider Patrick's paper with particular reference to his development for the circular case.

We define the linear Markov process by

$$x_s = \rho\, x_{s-1} + e_s$$

for all s, where the real constant ρ is called the serial correlation coefficient and the $[\, e_s \,]$ are independently and identically distributed, $N(0,1)$, random variables. The process is called circular if

$$x_s = \rho\, x_{s-1} + e_s$$

for $s = 1, \ldots, n$, where $x_o = x_n$ and $|\rho| < 1$.

Let $x_1, \ldots, x_n$ be a sample of observations from the circular process. They have a joint multivariate normal distribution since $e_1, \ldots, e_n$ are independent, $N(0, 1)$, random variables.

This distribution is given by

$$(2.1) \qquad dF(x_1, \ldots, x_n) = \frac{(1-\rho^n)}{(2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2}\{(1+\rho^2)(x_1^2 + \ldots + x_n^2) - 2\rho\,(x_1 x_2 + x_2 x_3 + \ldots + x_{n-1} x_n + x_n x_1)\}\right] dx_1 \ldots dx_n ,$$

which may be written in matrix notation as

$$(2.2) \qquad dF(x_1, \ldots, x_n) = \frac{(1-\rho^n)}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\, \underline{x}'\, \Sigma^{-1}\, \underline{x}\right] dx_1 \ldots dx_n ,$$

where,

$$\underline{x}' = (x_1 , \ldots , x_n)$$

and

$$\underline{\Sigma}^{-1} = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 & -\rho \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\rho & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\rho & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ -\rho & 0 & 0 & \cdots & 0 & -\rho & 1+\rho^2 \end{bmatrix}$$

with

$$\underline{|\Sigma^{-1}|}^{\frac{1}{2}} = 1 - \rho^n \quad .$$

As an estimate of ρ , we take the function

$$\hat{\rho}(x_1 , \ldots , x_n) = \frac{x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1}{x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + x_n^2} \tag{2.3}$$

which in matrix notation is given by

$$\hat{\rho} = \frac{\underline{x}' \ \underline{N} \ \underline{x}}{\underline{x}' \ \underline{D} \ \underline{x}} \tag{2.4}$$

where,

$$\underline{N} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

and

$$\underline{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\underline{N}$ is an indefinite symmetric matrix of constant coefficients, $\underline{D}$ is a positive definite symmetric matrix of constant coefficients, and $|\hat{\rho}| \leq 1$.

If we denote the distribution function of $\hat{\rho}$ by $H(\cdot)$, then by definition

$$(2.5) \qquad H(r) = P_r[\hat{\rho} \leq r] = P_r\left[\frac{\underline{x}'\underline{N}\,\underline{x}}{\underline{x}'\underline{D}\,\underline{x}} \leq r\right],$$

where r is any real number such that $|r| \leq 1$. In the following discussion, for reasons that will become apparent, r is restricted so that $|r| < 1$. Thus,

$$P_r\left[\frac{\underline{x}'\underline{N}\,\underline{x}}{\underline{x}'\underline{D}\,\underline{x}} \leq r\right] = \Pr[(\underline{x}'\underline{N}\,\underline{x}) \leq r(\underline{x}'\underline{D}\,\underline{x})] = P_r[\underline{x}'(\underline{N} - r\underline{D})\underline{x} \leq 0].$$

Putting

$$\underline{x}'(\underline{N} - r\underline{D})\underline{x} = u(x_1, \ldots, x_n)$$

and denoting the distribution function of u by $G(\cdot)$, we have

$$(2.6) \qquad H(r) = \Pr[\hat{\rho} \leq r] = P_r[u \leq 0] = G(0).$$

The matrix $(\underline{N} - r\underline{D})$ is indefinite symmetric with variable coefficients.

The characteristic function of $u(x_1, \ldots, x_n)$ is

$$(2.7) \qquad \varphi_u(\zeta) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[i\zeta u(x_1, \ldots, x_n)]\, dF(x_1, \ldots, x_n),$$

where ζ is a complex variable, and $u(x_1, \ldots, x_n)$ and $dF(x_1, \ldots, x_n)$ are defined as before.

We determine the distribution function of u from its characteristic function by considering the inversion formula

$$(2.8) \qquad G(v) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\exp[-i\zeta v]\, \varphi u(\zeta)\, d\zeta}{\zeta},$$

which, for $v = 0$, gives the distribution function of $\hat{\rho}$, that is,

$$H(r) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\varphi_u(\zeta)}{\zeta}\, d\zeta \,, \tag{2.9}$$

where

$$\oint = \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T} \right) \tag{2.10}$$

the path of integration being the line $\operatorname{Im}\zeta = 0$.

In matrix notation the characteristic function is

$$\begin{aligned} \varphi_u(\zeta) &= \frac{|\underline{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\,[i\zeta \underline{x}'(\underline{N} - r\underline{D})\,\underline{x}] \\ &\quad \times \exp\,[-\tfrac{1}{2}\,\underline{x}'\,\underline{\Sigma}^{-1}\,\underline{x}]\, dx_1 \cdots dx_n \\ &= \frac{|\underline{\Sigma}|^{-\frac{1}{2}}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\,[-\tfrac{1}{2}\,\underline{x}'\,\{\underline{\Sigma}^{-1} - 2i\zeta(\underline{N} - r\underline{D})\}\underline{x}\,] \\ &\quad dx_1 \cdots dx_n \,. \end{aligned} \tag{2.11}$$

There is an orthogonal transformation that will diagonalize $\{\underline{\Sigma}^{-1} - 2i\zeta(\underline{N} - r\underline{D})\}$.
Performing this transformation and integrating we get

$$\varphi_u(\zeta) = \frac{|\underline{\Sigma}|^{-\frac{1}{2}}}{|\underline{\Sigma}^{-1} - 2i\zeta(\underline{N} - r\underline{D})|^{\frac{1}{2}}} \tag{2.12}$$

where

$$|\underline{\Sigma}^{-1} - 2i\zeta(\underline{N} - r\underline{D})|^{-\frac{1}{2}}$$

is the Jacobian of the transformation.

By equation (2.14) we have

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$(2.13)\qquad \underline{N} - r\underline{D} = \begin{bmatrix} -r & \frac{1}{2} & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & -r & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -r & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -r & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -r & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} & -r \end{bmatrix} .$$

Thus

$$(2.14)\qquad \left|\underline{\Sigma}^{-1} - 2i\zeta\,(\underline{N} - r\underline{D})\right|$$

$$= \begin{vmatrix} 1+\rho^2+2ir\zeta & -(\rho+i\zeta) & 0 & \cdots & 0 & 0 & -(\rho+i\zeta) \\ -(\rho+i\zeta) & 1+\rho^2+2ir\zeta & -(\rho+i\zeta) & \cdots & 0 & 0 & 0 \\ 0 & -(\rho+i\zeta) & 1+\rho^2+2ir\zeta & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1+\rho^2+2ir\zeta & -(\rho+i\zeta) & 0 \\ 0 & 0 & 0 & \cdots & -(\rho+i\zeta) & 1+\rho^2+2ir\zeta & -(\rho+i\zeta) \\ -(\rho+i\zeta) & 0 & 0 & \cdots & 0 & -(\rho+i\zeta) & 1+\rho^2+2ir\zeta \end{vmatrix}$$

$$= (\rho+i\zeta)^n$$

$$\times \begin{vmatrix} \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} \end{vmatrix}$$

$$= (\rho+i\zeta)^n|\underline{B}| .$$

Thus, equation (2.12) becomes

$$(2.15)\qquad \varphi_u(\zeta) = \frac{1-\rho^n}{(\rho+i\zeta)^{n/2}}\,\frac{1}{|\underline{B}|^{1/2}} .$$

Evaluating $|\underline{B}|$ as a circulant we have

(2.16) $$|\underline{B}| = \frac{(1 - z^n)^2}{z^n}$$

where,

(2.17) $$z + \frac{1}{z} = \frac{1 + \rho^2 + 2ir\zeta}{\rho + i\zeta} .$$

Hence,

(2.18) $$\rho + i\zeta = z \frac{(1 - 2\rho r + \rho^2)}{(1 - 2rz + z^2)}$$

and

(2.19) $$\varphi_u(\zeta(z)) = \frac{1 - \rho^n}{(1 - 2\rho r + \rho^2)^{n/2}} \cdot \frac{(1 - 2rz + z^2)^{n/2}}{(1 - z^n)}$$

where,

(2.19a) $$\zeta(z) = i\rho - \frac{iz(1 - 2\rho r + \rho^2)}{1 - 2rz + z^2} .$$

Now,

(2.20) $$\frac{\zeta'(z)}{\zeta\ (z)} = \frac{(1 - 2\rho r + \rho^2)(1 - z^2)}{(1 - 2rz + z^2)(1 - \rho z)(z - \rho)} ,$$

thus, from equations (2.19) and (2.20) we have

(2.21) $$\frac{\varphi_u(\zeta(z))\ \zeta'(z)}{\zeta\ (z)} = \frac{1 - \rho^n}{(1 - 2\rho r + \rho^2)^{n/2 - 1}}$$

$$\times \quad \frac{(1 - z^2)(1 - 2rz + z^2)^{n/2 - 1}}{(z - \rho)(1 - \rho z)(1 - z^n)} .$$

Since we are prepared to tolerate errors of magnitude $O(\lambda^n)$ for some $|\lambda| < 1$, for neither r nor ρ near ± 1, we omit

$$\frac{1 - \rho^n}{1 - z^n} \quad \text{in} \quad (2.21) .$$

Thus, we have

$$(2.22) \qquad \frac{\varphi_u(\zeta(z))\,\zeta'(z)}{\zeta(z)} \sim \frac{(1-z^2)(1-2rz+z^2)^{n/2\,-1}}{(1-2\rho r+\rho^2)^{n/2\,-1}\,(z-\rho)(1-\rho z)} .$$

Now, equation (2.22) is of the form

$$(2.23) \qquad \frac{\varphi_u(\zeta(z))\,\zeta'(z)dz}{\zeta\,(z)} \sim \frac{K'}{(z-\rho)}\; g(z)(1-2rz+z^2)^s dz$$

where,

1) $\zeta = \zeta(z)$ represents the transformation (2.17) ,

2) K' is a function of (ρ, r),

3) $g(z)$ is an algebraic function taking real values for real z and is regular in a simply-connected subset of $|\,z\,| \leq 1$, which contains $z = \rho$ and $z = r$,

4) s is a real variable taking large values $[s = \frac{n}{2} - 1]$.

In equation (2.23) we have, for our case,

$$K' = (1 - 2\rho r + \rho^2)^{-\frac{n}{2}+1}$$

and

$$g(z) = \frac{(1-z^2)}{(1-\rho z)} .$$

The transformation (2.17) maps the ζ-plane cut from $-i\,\infty$ to $-i\,(1-\rho^2)/2(1-r)$ and from $i(1+\rho)^2/2(1+r)$ to $+i\,\infty$ onto the open disc $|\,z\,| < 1$, with the cut being mapped onto the unit circle $|\,z\,| = 1$.

If we denote the z-image of $\mathrm{Im}\zeta = 0$ by Γ , then Γ represents the transformed path of integration in the z-plane, with the end points $z = r - i\sqrt{1-r^2}$ and $z = r + i\sqrt{1-r^2}$ corresponding to the points $\zeta = -\infty$ and $\zeta = \infty$ respectively. The z-image of the point $\zeta = 0$ is the point $z = \rho$ at which the path Γ crosses the real axis. The curve Γ is symmetric about $\mathrm{Im}\ z = 0$, thus points below $\mathrm{Im}\ z = 0$ lying on Γ are complex

conjugates of points on the curve above the line $\operatorname{Im} z = 0$.

Thus the z-correspondent of the ζ-integral in the inversion formula (2.9) is defined to be

$$(2.24) \quad \frac{K'}{2\pi i} \fint_{[\Gamma]} \frac{1}{(z-\rho)} g(z)(1-2rz+z^2)^s dz$$

$$\left[= \frac{1}{2\pi i} \oint \frac{\varphi_u(\zeta)}{\zeta} d\zeta \right]$$

where,

$$(2.25) \quad \fint_{[\Gamma]} = \lim_{\substack{a \xrightarrow{\Gamma} \rho \\ z^* \xrightarrow{\Gamma} r+i\sqrt{1-r^2}}} \left[\int_{\bar{z}^*}^{\bar{a}} + \int_a^{z^*} \right] .$$

The integrals defined are curvilinear on Γ , with a and its complex conjugate $\bar{a}$ points on Γ as are z^* and $\bar{z}^*$. Thus the z-integral exists in the sense of a Cauchy Principal Value as defined in Appendix I .

Now, it is clear that

$$|g(z)| < \infty \quad \text{for} \quad z = r \pm i\sqrt{1-r^2} .$$

Thus

$$(2.26) \quad \lim_{z \to r + i\sqrt{1-r^2}} \frac{1}{(z-\rho)} g(z)(1-2rz+z^2)^s = 0 ,$$

and

$$(2.27) \quad \lim_{z \to r - i\sqrt{1-r^2}} \frac{1}{(z-\rho)} g(z)(1-2rz+z^2)^s = 0 .$$

Thus we have

$$(2.28) \quad \fint_{[\Gamma]} = \lim_{a \xrightarrow{\Gamma} p} \left[\int_{\bar{A}}^{\bar{a}} + \int_a^{A} \right]$$

where $\quad A = r + i\sqrt{1-r^2} \quad .$

We wish to evaluate the integral

$$(2.29) \quad \frac{K'}{2\pi i} \int_{[\Gamma]} \frac{1}{(z - \rho)} \; g(z)(1 - 2rz + z^2)^s \; dz \, ,$$

where,

$$\int_{[\Gamma]} = \lim_{a \xrightarrow{\Gamma} \rho} \left[\int_{\bar{A}}^{\bar{a}} + \int_{a}^{A} \right] , \quad (A = r + i\sqrt{1 - r^2})$$

and

$g(z)$ is regular in $|z| < 1$.

We deform the path of integration to become the straight line running from $z = r + i\sqrt{1 - r^2}$ to $z = r - i\sqrt{1 - r^2}$, and passing through the critical point at $z = r$. The deformed path is made in such a manner that

$$(2.30) \quad \left| 1 - 2rz + z^2 \right| \leq (1 - r^2)$$

for all z on the deformed path, which we denote by L .

The integral is regular for all z in $|z| < 1$, except at the point $z = \rho$, and therefore the integrand has only one residue term. This residue term being the one corresponding to the point $z = \rho$.

We have three cases for consideration. They are $r < \rho$, $r = \rho$, and $r > \rho$. At present we consider the two cases $r < \rho$ and $r > \rho$, dealing with the case $r = \rho$ at a later stage.

We now indicate the following.

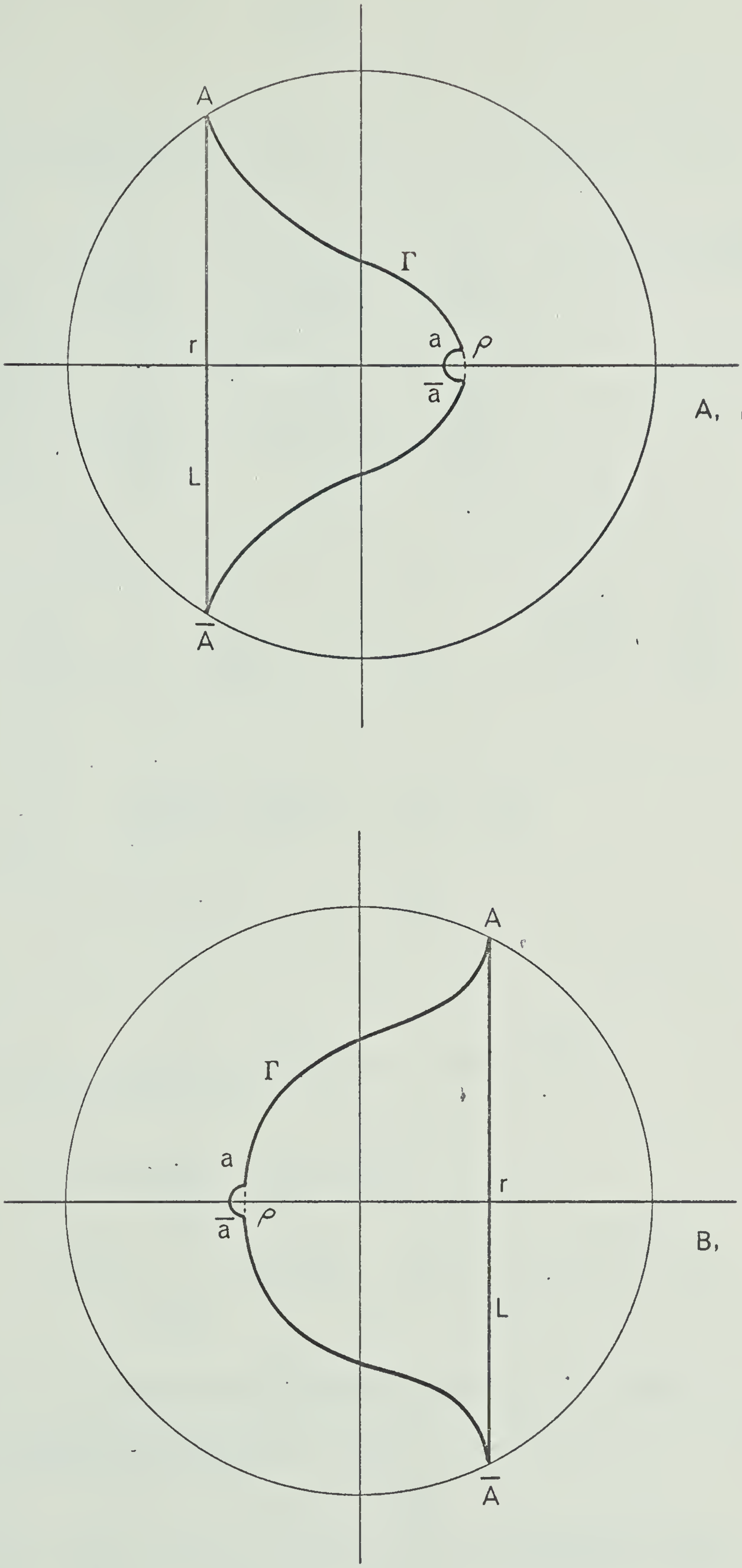
(i) Γ is the deformed path joining $\bar{A}\ \bar{a}\ \rho\ a\ A$. (as before)

(ii) L is the straight line joining $\bar{A}\ r\ A$.

(iii) C is the curve $\bar{a}\ a$ [the indentation around $z = \rho$] .

(iv) $\int_{x_1 \, [\;]}^{x_2}$, as before, represents the integral from x_1 to x_2 on the appropriate curve signified in [ ] .

For $r < \rho$, the line of integration is in the positive direction along the simple closed contour $\bar{A}\ \bar{a}\ a\ A\ r\ \bar{A}$, which excludes the point



A, $r < \rho$

B, $r > \rho$

Fig. 0

Admissable Deformed Paths of Integration

$z = \rho$ by the indentation C . Thus,

$$\int_{\bar{A}\,[\Gamma]}^{\bar{a}} + \int_{\bar{a}\,[C]}^{a} + \int_{a\,[\Gamma]}^{A} + \int_{A\,[L]}^{\bar{A}} = 0 \text{ , so that}$$

$$\int_{\bar{A}\,[\Gamma]}^{\bar{a}} + \int_{a\,[\Gamma]}^{A} = \int_{\bar{A}\,[L]}^{A} + \int_{a\,[C]}^{\bar{a}} .$$

Hence,

$$\lim_{a \xrightarrow{\Gamma} \rho} \left[\int_{\bar{A}\,[\Gamma]}^{\bar{a}} + \int_{a\,[\Gamma]}^{A} \right] = \int_{\bar{A}\,[L]}^{A} + \lim_{a \xrightarrow{C} \rho} \int_{a\,[C]}^{\bar{a}} .$$

Now,

$$\lim_{a \xrightarrow{C} \rho} \int_{a}^{\bar{a}} \frac{1}{(z - \rho)} g(z)(1 - 2rz + z^2)^s$$

$$= \pi i \, g(\rho)(1 - 2\rho r + \rho^2)^s ,$$

and therefore,

$$(2.31) \qquad \frac{K'}{2\pi i} \int_{[\Gamma]} \frac{1}{(z - \rho)} g\,(z)(1 - 2rz + z^2)^s \, dz$$

$$= \frac{K'}{2} g(\rho)(1 - 2\rho r + \rho^2)^s$$

$$+ \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z - \rho)} g(z)(1 - 2rz + z^2)^s \, dz .$$

Now for $r > \rho$, the contour encloses the pole $z = \rho$, and the integral is taken in the negative direction. Thus

$$\int_{\bar{A}\,[\Gamma]}^{\bar{a}} + \int_{\bar{a}\,[C]}^{a} + \int_{a\,[\Gamma]}^{A} + \int_{A\,[L]}^{\bar{A}} = -2\pi i g(\rho)(1 - 2\rho r + \rho^2)^s ,$$

so that

$$\int_{\bar{A}}^{\bar{a}}_{[\Gamma]} + \int_{a}^{A}_{[\Gamma]} = \int_{\bar{A}}^{A}_{[L]} + \int_{a}^{\bar{a}}_{[C]} - 2\pi i g(\rho)(1 - 2\rho r + \rho^2)^s.$$

Thus

$$(2.32) \qquad \frac{K'}{2\pi i} \fint_{[\Gamma]} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz$$

$$= \frac{-K'}{2} g(\rho)(1 - 2\rho r + \rho^2)^s$$

$$+ \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz .$$

By PATRICK (1964) , Appendix II

$$K' g(\rho)(1 - 2\rho r + \rho^2)^s = 1$$

so that, for $r < \rho$, (2.31) becomes

$$(2.33) \qquad \frac{K'}{2\pi i} \fint_{[\Gamma]} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz$$

$$= \frac{1}{2} + \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz$$

and, for $r > \rho$, (2.32) becomes

$$(2.34) \qquad \frac{K'}{2\pi i} \fint_{[\Gamma]} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz$$

$$= -\frac{1}{2} + \frac{1}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz .$$

Thus the inversion formula (2.9) (using (2.24)) for $r < \rho$ becomes

$$(2.35) \qquad H(r) = \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(\rho-z)} g(z)(1 - 2rz + z^2)^s \, dz ,$$

and, for $r > \rho$ becomes

$$(2.36) \qquad H(r) = 1 - \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z-\rho)} g(z)(1 - 2rz + z^2)^s \, dz \, ,$$

where L is the straight line from $\bar{A}$ to A .

The integral

$$(2.37) \qquad \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(\rho-z)} g(z)(1 - 2rz + z^2)^s \, dz$$

where, L is the line from $z = \bar{A} = r - i\sqrt{1 - r^2}$ to $z = A = r + i\sqrt{1 - r^2}$ passing through $z = r$, has a critical point at $z = r$. We change the variable of integration so that in terms of the new variable of integration the critical point is at the origin. We use the linear transformation

$$(2.38) \qquad z = r + it\sqrt{1 - r^2}$$

so that,

$$(2.39) \qquad dz = i\sqrt{1 - r^2} \, dt \, ,$$

$$(2.40) \qquad (\rho - z) = \rho - r - it\sqrt{1 - r^2} \, ,$$

$$(2.41) \qquad (1 - 2rz + z^2) = (1 - r^2)(1 - t^2) \, ,$$

$$(2.42) \qquad g(z) = g(r + it\sqrt{1 - r^2}) = \chi(t) \, ,$$

and L^* , the t-image of L , is the straight line from $t = -1$ to $t + 1$ passing through $t = 0$ [i.e., a segment of $\operatorname{Im} t = 0$] .

Thus, the t-integrals corresponding to (2.35) and (2.36) are, for $r < \rho$

$$(2.43) \qquad H(r) = \frac{K'(1 - r^2)^{s+\frac{1}{2}}}{2\pi} \int_{-1\,[L^*]}^{+1} \frac{1}{(\rho - r - it\sqrt{1 - r^2})} \chi(t)(1 - t^2)^s \, dt$$

and, for $r > \rho$

$$(2.44) \qquad H(r) = 1 - \frac{K'(1 - r^2)^{s+\frac{1}{2}}}{2\pi} \int_{\substack{-1 \\ [L^*]}}^{+1} \frac{1}{(r - \rho + it\sqrt{1 - r^2})} \chi(t)(1 - t^2)^s dt \, .$$

We, therefore, seek an asymptotic representation for the integral

$$(2.45) \qquad \int_{\substack{-1 \\ [L^*]}}^{+1} \frac{\chi(t)}{(\rho - r - it\sqrt{1 - r^2})} (1 - t^2)^s \, dt$$

where,

(i) L^* is the line $\mathrm{Im} t = 0$,

(ii) s is a real variable assuming large values,

(iii) ρ , r are real variables, ρ fixed such that $|\rho| < 1$, and r varying continuously over the set $\{\xi : |\xi| < 1 \text{ and } \xi \neq \rho\}$,

(iv) $\chi(t)$ is an analytic function of t regular in some neighbourhood of $t = 0$ and $|\chi(t)| < \infty$ for all t in $[-1 , +1]$, i.e. on the path of integration.

Assuming the results obtained by PATRICK (1964) , Appendix III, we have

$$(2.46) \qquad \int_{\substack{-1 \\ [L^*]}}^{+1} \frac{\chi(t)}{\rho - r - it\sqrt{1 - r^2}} (1 - t^2)^s \, dt$$

$$= \frac{\rho - r}{1 - r^2} \sum_{m=0}^{M} \sum_{k=0}^{m} C_k^{(m,1)} \Gamma(m+k+\tfrac{1}{2}) \left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{m+k-\frac{1}{2}}$$

$$\times \; s^k \, U\left(m + k + \tfrac{1}{2} \, ; \, m + k + \tfrac{1}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right)$$

$$- \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{(m-1,2)} \Gamma(m + k + \tfrac{1}{2}) \left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{m+k-\frac{1}{2}}$$

$$\times \; s^k \, U\left(m + k + \tfrac{1}{2} \, ; \, m + k + \tfrac{1}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right)$$

$$+ \; 0 \left[\left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{M+\frac{1}{2}} \Gamma(M + \tfrac{3}{2}) \, U \left(M + \tfrac{3}{2} \, ; \, M + \tfrac{3}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right) \right]$$

where, $U(\; ; \; ; \;)$ is the Confluent Hypergeometric function in the notation of SLATER (1960), and the $C_k^{(m,1)}$, $C_k^{(m-1,2)}$ are given by

$$(2.47) \qquad \sum_{m=0}^{\infty} \sum_{k=0}^{M} C_k^{(m,1)} \; s^k \; t^{2(m+k)} = \frac{1}{2} [\chi(t) + \chi(-t)]$$

$$\times \; \exp \, [st^2 + s \log (1 - t^2) \,]$$

and,

$$(2.48) \qquad \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} C_k^{(m-1,2)} \; s^k \; t^{2(m+k-1)}$$

$$= - \frac{i}{2\sqrt{1 - r^2}} \; [\chi(t) - \chi(-t)] \exp \, [st^2 + s \log (1 - t^2)] \, .$$

Now, by equations (2.44) and (2.45), we have, for $r < \rho$,

$$(2.49) \qquad H(r) = \frac{K'(1 - r^2)^s}{2\pi} \sum_{m=0}^{M} \sum_{k=0}^{m} C_k^{(m,1)} \; \Gamma \, (m + k + \tfrac{1}{2})$$

$$\times \left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{m+k} s^k \, U \left(m + k + \tfrac{1}{2} ; \, m + k + \tfrac{1}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right)$$

$$- \frac{K'(1 - r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{(m-1,2)} \; \Gamma \, (m + k + \tfrac{1}{2})$$

$$\times \left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{m+k-\frac{1}{2}} s^k \, U \left(m + k + \tfrac{1}{2} \, ; \, m + k + \tfrac{1}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right)$$

$$+ \; 0 \left[\left\{ \frac{(\rho - r)^2}{1 - r^2} \right\}^{M+\frac{1}{2}} \Gamma \, (M + \tfrac{3}{2}) \, U \left(M + \tfrac{3}{2} \, ; \, M + \tfrac{3}{2} \, ; \, \frac{s(\rho - r)^2}{1 - r^2} \right) \right]$$

and, for $r > \rho$,

$$(2.50) \qquad H(r) = 1 - \frac{K'(1 - r^2)^s}{2\pi} \sum_{m=0}^{M} \sum_{k=0}^{m} C_k^{(m,1)} \, \Gamma \, (m + k + \tfrac{1}{2})$$

$$\times \left\{ \frac{(r-\rho)^2}{1-r^2} \right\}^{m+k} s^k \, U\left(m+k+\tfrac{1}{2} \; ; \; m+k+\tfrac{1}{2} \; ; \; \frac{s(r-\rho)^\rho}{1-r^2} \right)$$

$$- \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{(m-1,2)} \Gamma(m+k+\tfrac{1}{2}) \left\{ \frac{(r-\rho)^2}{1-\rho^2} \right\}^{m+k-\frac{1}{2}} s^k$$

$$\times U\left(m+k+\tfrac{1}{2}; \; m+k+\tfrac{1}{2}; \; \frac{s(r-\rho)^2}{1-r^2} \right) + O\left[\left\{ \frac{(r-\rho)^2}{1-r^2} \right\}^{M+\frac{1}{2}} \Gamma(M+\tfrac{3}{2}) \right.$$

$$\left. \times \; U\left(M+\tfrac{3}{2} \; ; \; M+\tfrac{3}{2} \; ; \; \frac{s(r-\rho)^2}{1-r^2} \right) \right] .$$

Now,

$$\text{(2.51)} \quad \Gamma(\sigma) \left\{ \frac{(\rho-r)^2}{1-r^2} \right\}^{\sigma-1} U\left(\sigma \; ; \; \sigma \; ; \; \frac{s(\rho-r)^2}{1-r^2} \right) = \int_0^\infty \frac{\nu^{\rho-1}}{\left\{ \frac{(\rho-r)^2}{1-r^2} + \nu \right\}} \exp[-s\nu] \, d\nu,$$

and

$$\text{(2.52)} \quad U\left(\tfrac{1}{2} \; ; \; \tfrac{1}{2} \; ; \; \frac{s(\rho-r)^2}{1-r^2} \right) = \exp \frac{s(\rho-r)^2}{1-r^2} \int_{\frac{s(\rho-r)^2}{1-r^2}}^\infty \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu .$$

Using (2.51) for $m+k>0$ and (2.52) for $m+k=0$, we get, in terms of the above integrals, for $r < \rho$,

$$\text{(2.53)} \quad H(r) = \frac{K'(1-r^2)^s}{2\pi} C_o^{(0,1)} \Gamma(\tfrac{1}{2}) \exp\left[\frac{s(\rho-r)^2}{1-r^2} \right] \int_{\frac{s(\rho-r)^2}{1-r^2}}^\infty \nu^{-\frac{1}{2}} \exp[-v] \, d\nu$$

$$+ \frac{K'(1-r^2)^{s-\frac{1}{2}}(\rho-r)}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m} C_k^{(m,1)} s^k \left[\int_0^\infty \frac{\nu^{m+k-\frac{1}{2}}}{\left\{ \frac{(\rho-r)^2}{1-r^2} + \nu \right\}} \exp[-s\nu] \, d\nu \right]$$

$$- \frac{K'(1-r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{(m-1,2)} s^k \left[\int_0^\infty \frac{\nu^{m+k-\frac{1}{2}}}{\left\{ \frac{(\rho-r)^2}{1-r^2} + \nu \right\}} \exp[-s\nu] \, d\nu \right]$$

$$+ O\left(\int_0^\infty \frac{\nu^{M+\frac{1}{2}}}{\left\{ \frac{(\rho-r)^2}{1-r^2} \right\}} \exp[-s\nu] \, d\nu \right) ,$$

and, for $r > \rho$,

(2.54) $$H(r) = 1 - \frac{K'(1 - r^2)^s}{2\pi} \; C_o^{(0,1)} \, \Gamma\left(\tfrac{1}{2}\right)$$

$$\times \exp\left[\frac{s(r-\rho)^2}{1 - r^2}\right] \int_{\frac{s(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu$$

$$- \frac{K'(1 - r^2)^{s-\frac{1}{2}} (r - \rho)}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m} C_k^{(m,1)} \, s^k$$

$$\times \left[\int_0^{\infty} \frac{\nu^{m+k-\frac{1}{2}}}{\left\{\frac{(r-\rho)^2}{1-r^2} + \nu\right\}} \exp[-s\nu] \, d\nu \right]$$

$$- \frac{K'(1 - r^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{(m-1,2)} \, s^k$$

$$\times \left[\int_0^{\infty} \frac{\nu^{m+k-\frac{1}{2}}}{\left\{\frac{(r-\rho)^2}{1-r^2} + \nu\right\}} \exp[-s\nu] \, d\nu \right]$$

$$+ \, O\left(\int_0^{\infty} \frac{\nu^{M+\frac{1}{2}}}{\left\{\frac{(r-\rho)^2}{1-r^2} + \nu\right\}} \exp[-s\nu] \, d\nu \right).$$

Thus, as $r \to \rho$, (2.53) becomes

(2.55) $$H(\rho - 0) = \frac{K^*(1 - \rho^2)^s \; C_o^* \, (0,1)}{2}$$

$$- \frac{K^*(1 - \rho^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{*(m-1,2)} \; \frac{\Gamma\left(m + k - \frac{1}{2}\right)}{s^{m-\frac{1}{2}}}$$

$$+ \; O\left(\frac{1}{s^{M+\frac{1}{2}}}\right)$$

and (2.54) becomes

$$(2.56) \qquad H(\rho + 0) = 1 - \frac{K^* (1 - \rho^2)^s \, C_o^{*(0,1)}}{2}$$

$$- \frac{K^*(1 - \rho^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{*(m-1,2)} \frac{\Gamma(m + k - \frac{1}{2})}{s^{m-\frac{1}{2}}}$$

$$+ \; 0\left(\frac{1}{s^{M+\frac{1}{2}}} \right)$$

where,

$$(2.57) \qquad K^* = K' \quad \text{at} \quad r = \rho \; .$$

$$(2.58) \qquad C_k^{*(,)} = C_k^{(,)} \quad \text{at} \quad r = \rho \; ,$$

and the interchange of limits and integration is justified since the integrals are uniformly convergent.

Now, by (2.47) ,

$$(2.59) \qquad C_o^{*(0,1)} = \chi^* (0)$$

where, by equation (2.42)

$$\chi^* (0) = g(\rho)$$

and

$$(2.60) \qquad \chi(t) = \chi^* (t) \quad \text{at} \quad r = \rho \; .$$

By PATRICK (1964), Appendix II, (equation 15)

$$(2.61) \qquad \frac{K^* (1 - \rho^2)^s \; \chi^* (0)}{2} = \frac{K^* (1 - \rho^2)^s \; g(\rho)}{2} = \frac{1}{2} \; .$$

Thus,

$$(2.62) \qquad H(\rho - 0) = H(\rho + 0) = H(\rho)$$

$$= \frac{1}{2} - \frac{K^*(1 - \rho^2)^{s+\frac{1}{2}}}{2\pi} \sum_{m=1}^{M} \sum_{k=0}^{m-1} C_k^{*(m-1,2)} \frac{\Gamma(m + k - \frac{1}{2})}{s^{m-\frac{1}{2}}}$$

$$+ 0 \left(\frac{1}{s^{M+\frac{1}{2}}} \right) .$$

We take equation (2.62) to be the asymptotic value of $H(r)$ at $r = \rho$.

Thus, in equations (2.53) and (2.54) for $M = 0$, we have the following asymptotic expressions, for $r < \rho$

$$(2.63) \qquad H(r) = \frac{K'(1 - r^2)^s}{2\pi} \chi(0) \Gamma(\tfrac{1}{2})$$

$$\times \exp \left[\frac{s(\rho - r)^2}{1 - r^2} \right] \int_{\frac{s(\rho-r)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu$$

$$+ 0 \left(\int_0^{\infty} \frac{\nu^{\frac{1}{2}}}{\left\{ \frac{(\rho - r)^2}{1 - r^2} + \nu \right\}} \exp[-s\nu] \, d\nu \right)$$

and, for $r > \rho$

$$(2.64) \qquad H(r) = 1 - \frac{K'(1 - r^2)^s}{2\pi} \chi(0) \Gamma(\tfrac{1}{2})$$

$$\times \exp \left[\frac{s(r - \rho)^2}{1 - r^2} \right] \int_{\frac{s(r - \rho)^2}{1 - r^2}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu$$

$$+ 0 \left(\int_0^{\infty} \frac{\nu^{\frac{1}{2}}}{\left\{ \frac{(r - \rho)^2}{1 - r^2} + \nu \right\}} \exp[-s\nu] \, d\nu \right)$$

which, for $r = \rho$, give

$$(2.65) \qquad H(\rho) = \tfrac{1}{2} + 0 \left(\frac{1}{s^{\frac{1}{2}}} \right)$$

which would be an approximation to the first term in (2.62) . Thus, we take as the asymptotic representation for $r \leq \rho$,

$$(2.66)\quad H(r) \sim \frac{K'(1-r^2)^s\,\chi(0)}{2\pi}\,\Gamma(\tfrac{1}{2})\,\exp\left[\frac{s(\rho-r)^2}{1-r^2}\right]\int_{\frac{s(\rho-r)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp[-\nu]\,d\nu$$

and, for $r \geq \rho$,

$$(2.67)\quad H(r) \sim 1 - \frac{K'(1-r^2)^s\,\chi(0)}{2\pi}\,\Gamma(\tfrac{1}{2})\,\exp\left[\frac{s(r-\rho)^2}{1-r^2}\right]\int_{\frac{s(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp[-\nu]\,d\nu \,.$$

Equation (2.42) yields

$$(2.68)\qquad \chi(0) = g(r)\,.$$

Thus, corresponding to equations (2.35) and (2.36) , we have for $r \leq \rho$,

$$(2.69)\quad H(r) \sim \frac{K'(1-r^2)^s\,g(r)}{2\pi}\,\Gamma(\tfrac{1}{2})\,\exp\left[\frac{s(r-\rho)^2}{1-r^2}\right]\int_{\frac{s(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp[-\nu]\,d\nu\,,$$

and, for $r \geq \rho$

$$(2.70)\quad H(r) \sim 1 - \frac{K'(1-r^2)^s g(r)}{2\pi}\,\Gamma(\tfrac{1}{2})\,\exp\left[\frac{s(r-\rho)^2}{1-r^2}\right]\int_{\frac{s(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp[-\nu]\,d\nu\,.$$

Now, by equations (2.22) and (2.23) we have

$$(2.71)\qquad \text{(i)}\quad s = \frac{n}{2} - 1$$

$$\text{(ii)}\quad K' = \frac{1}{(1-2\rho r+\rho^2)^{\frac{n}{2}-1}}$$

$$\text{(iii)}\quad g(z) = \frac{1-z^2}{1-\rho z}$$

Thus equations (2.69) and (2.70) give, for $r \leq \rho$

$$(2.72) \qquad H(r) \sim \frac{1}{2\sqrt{\pi}} \frac{(1 - r^2)^{\frac{n}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2} - 1}} \frac{\exp\left[(\frac{n}{2} - 1) \frac{(\rho - r)^2}{1 - r^2} \right]}{(1 - \rho r)}$$

$$\times \int_{\left\{(\frac{n}{2} - 1) \frac{(\rho - r)^2}{1 - r^2}\right\}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu$$

and, for $r \geq \rho$

$$(2.73) \qquad H(r) \sim 1 - \frac{1}{2\sqrt{\pi}} \frac{(1 - r^2)^{\frac{n}{2}}}{(1 - 2\rho r + \rho^2)^{\frac{n}{2} - 1}} \frac{\exp\left[(\frac{n}{2} - 1) \frac{(r - \rho)^2}{1 - r^2} \right]}{(1 - \rho r)}$$

$$\times \int_{\left\{(\frac{n}{2} - 1) \frac{(r - \rho)^2}{1 - r^2}\right\}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu$$

and, in particular, for $r = \rho$,

$$(2.74) \qquad H(\rho) \sim \tfrac{1}{2} .$$

The magnitude of error incurred by the above approximation is

$$(2.75) \qquad R = O\left(\frac{1}{n^{\frac{1}{2}}} \right) \quad \text{as} \quad n \to \infty .$$

This order relation being uniform in r and n .

Now, for the case $r \leq \rho$, consider

$$(2.76) \qquad \exp\left[\left(\frac{n}{2} - 1 \right) \frac{(\rho - r)^2}{1 - r^2} \right] \int_{\left\{\left(\frac{n}{2} - 1 \right) \frac{(\rho - r)^2}{1 - r^2}\right\}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu] \, d\nu .$$

Put $\nu = \frac{1}{2} t^2$.

Then,

$$d\nu = t\,dt ,$$

$$\nu^{-\frac{1}{2}} = \sqrt{2}/t$$

and,

$$\nu = \left(\frac{n}{2} - 1\right)\frac{(\rho - r)^2}{1 - r^2} \quad \text{corresponds to} \quad t = \frac{\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}} \quad .$$

Thus, (2.76) becomes

$$(2.77) \qquad \sqrt{2}\int_{\left\{\frac{\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right\}}^{\infty} \exp\,[-\tfrac{1}{2}t^2\,]\,dt \Big/ \exp\left[-\tfrac{1}{2}\left\{\frac{\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right\}^2\right]$$

$$= \sqrt{2}\int_{-\infty}^{\left\{\frac{-\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right\}} \exp\,[-\tfrac{1}{2}\,t^2]\,dt \Big/ \exp\left[-\tfrac{1}{2}\left\{\frac{\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right\}^2\right]$$

$$= \sqrt{2}\;\frac{\Phi\left[\frac{-\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right]}{\varphi\left[\frac{-\sqrt{n-2}\,(\rho - r)}{\sqrt{1 - r^2}}\right]}$$

where Φ ( ) is the standardized normal distribution function and φ ( ) the standard normal probability density function.

The corresponding result for $r \geq \rho$ is

$$(2.78) \qquad \sqrt{2}\;\frac{\Phi\left[\frac{-\sqrt{n-2}\,(r - \rho)}{\sqrt{1 - r^2}}\right]}{\varphi\left[\frac{-\sqrt{n-2}\,(r - \rho)}{\sqrt{1 - r^2}}\right]} \quad .$$

Thus, by equations (2.72) and (2.73) the approximate distribution function for $\hat{\rho}$ is given, for $r \leq \rho$, by

$$(2.79) \qquad H(r) \sim \frac{1}{\sqrt{2\pi}}\;\frac{(1 - r^2)^{\frac{n}{2}}}{(1 - \rho r)(1 - 2\rho r + \rho^2)^{\frac{n}{2}-1}}\;\frac{\Phi\left[\frac{-\sqrt{n-2}\,|\rho - r|}{\sqrt{1 - r^2}}\right]}{\varphi\left[\frac{-\sqrt{n-2}\,|\rho - r|}{\sqrt{1 - r^2}}\right]}$$

and, for $r \geq \rho$

$$H(r) \sim 1 - \frac{1}{\sqrt{2\pi}} \frac{(1 - r^2)^{\frac{n}{2}}}{(1 - \rho r)(1 - 2\rho r + \rho^2)^{\frac{n}{2} - 1}} \frac{\Phi\left[\dfrac{-\sqrt{n-2}\,|\rho - r|}{\sqrt{1 - r^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-2}\,|\rho - r|}{\sqrt{1 - r^2}}\right]} \, . \tag{2.80}$$

CHAPTER III

THE APPROXIMATE DISTRIBUTION FUNCTION OF AN ESTIMATE OF THE SERIAL CORRELATION COEFFICIENT IN A STATIONARY LINEAR MARKOV PROCESS

For the linear Markov process defined in Chapter II, we derive an approximate expression for the distribution function of a sample estimate of the serial correlation coefficient ρ with unknown mean in the stationary case. Our sample estimate $\hat{\rho}$ of ρ is given by (1.1) and (1.2), for which the approximate distribution was obtained by PATTON (1961).

We assume the linear Markov process to be stationary, that is,

$$x_s = \rho\, x_{s-1} + e_s \tag{3.1}$$

for $s = 1, \ldots, n$, where $|\rho| < 1$ and

$$\operatorname{cov}(x_s, x_{s-\tau}) = \frac{\rho^{|\tau|}}{1-\rho^2} .$$

Let $x_1, \ldots, x_n$ be a sample of observations from the stationary process. They have a joint multivariate normal distribution since $e_1, \ldots, e_n$ are independent, $N(0, 1)$, random variables.

This distribution is given by

$$dF(x_1, \ldots, x_n) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\{-\tfrac{1}{2}[x_1^2 + (1+\rho^2)(x_1^2 + \ldots + x_n^2) + x_n^2 - 2\rho(x_1x_2 + \ldots + x_{n-1}x_n)]\}\, dx_1 \ldots dx_n , \tag{3.2}$$

which, in matrix notation may be written as

(3.3) $$dF\,(x_1, \ldots, x_n) = \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi)^{n/2}} \exp\,[-\tfrac{1}{2}\,\underline{x}'\,\underline{\Sigma}^{-1}\,\underline{x}\,]\,dx_1 \ldots dx_n\ ,$$

where

$$\underline{x}' = (x_1\,, \ldots, x_n)\ ,$$

and

$$\underline{\Sigma}^{-1} = \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\rho & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\rho & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}$$

and

$$|\,\underline{\Sigma}\,|^{-\frac{1}{2}} = (1-\rho^2)^{\frac{1}{2}}\ .$$

When the mean is unknown we take the ratio

(3.4) $$\hat{\rho} = \frac{C}{C_o}$$, as an estimate of ρ , where ,

(3.5) $$\begin{cases} C = \frac{1}{2}(x_1-\bar{x})^2 + (x_1-\bar{x})(x_2-\bar{x}) + \ldots + (x_{n-1}-\bar{x})(x_n-\bar{x}) \\ \quad + \frac{1}{2}(x_n-\bar{x})^2 \\ \quad = c - n\bar{x}^2\ , \\ C_o = (x_1-\bar{x})^2 + \ldots + (x_n-\bar{x})^2 = c_o - n\bar{x}^2\ , \end{cases}$$

with

$$c = \tfrac{1}{2}\, x_1^2 + x_1 x_2 + \ldots + x_{n-1} x_n + \tfrac{1}{2}\, x_n^2 \qquad ,$$

$$c_o = x_1^2 + \ldots + x_n^2 \qquad ,$$

and,

$$\bar{x} = \frac{x_1 + \ldots + x_n}{n} \qquad .$$

In matrix notation

$$\hat{\rho} = \frac{\underline{x}'\underline{N}\ \underline{x}}{\underline{x}'\underline{D}\ \underline{x}} \tag{3.6}$$

where,

$$\underline{N} = \begin{bmatrix}
(\frac{1}{2}-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n}) & & & & & & (-\frac{1}{n}) \\
(\frac{1}{2}-\frac{1}{n}) & (-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n}) & & & & & \\
 & (\frac{1}{2}-\frac{1}{n}) & (-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n}) & & & & \\
 & & \cdot & \cdot & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & \cdot & \cdot & \\
 & & & & (\frac{1}{2}-\frac{1}{n}) & (-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n}) & \\
 & & & & & (\frac{1}{2}-\frac{1}{n}) & (-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n}) \\
(-\frac{1}{n}) & & & & & & (\frac{1}{2}-\frac{1}{n}) & (\frac{1}{2}-\frac{1}{n})
\end{bmatrix}$$

and

$$\underset{\sim}{D} = \begin{bmatrix} (1-\frac{1}{n}) & & & & & & (-\frac{1}{n}) \\ & (1-\frac{1}{n}) & & & & & \\ & & (1-\frac{1}{n}) & & & & \\ & & & \ddots & & & \\ & & & & (1-\frac{1}{n}) & & \\ & & & & & (1-\frac{1}{n}) & \\ (-\frac{1}{n}) & & & & & & (1-\frac{1}{n}) \end{bmatrix}$$

with

$$|\hat{\rho}| \leq 1 .$$

Denoting the distribution function of $\hat{\rho}$ by $H(\cdot)$, the following results of Chapter II are again valid.

(3.7) (i) $H(r) = P_r[\hat{\rho} \leq r] = P_r[u \leq 0] = G(0)$, where $G(\cdot)$ denotes the distribution function of $u(x_1, \ldots, x_n) = \underset{\sim}{x}'(\underset{\sim}{N} - r\underset{\sim}{D})\underset{\sim}{x}$.

(ii) The characteristic function of $u(x_1, \ldots, x_n)$ is

$$\varphi_u(\zeta) = \frac{|\underset{\sim}{\Sigma}|^{-\frac{1}{2}}}{|\underset{\sim}{\Sigma}^{-1} - 2i\zeta(\underset{\sim}{N} - r\underset{\sim}{D})|^{\frac{1}{2}}}$$

where

$$|\underset{\sim}{\Sigma}^{-1} - 2i\zeta(\underset{\sim}{N} - r\underset{\sim}{D})|^{-\frac{1}{2}}$$

is the Jacobian of the transformation.

(iii) The distribution function of $\hat{\rho}$ is

$$H(r) = \frac{1}{2} - \frac{1}{2\pi i} \oint \frac{\varphi_u(\zeta)}{\zeta}\, d\zeta ,$$

where,

$$\oint = \lim_{\substack{T \to \infty \\ \epsilon \to 0}} \left(\int_{-T}^{-\epsilon} + \int_{\epsilon}^{T} \right),$$

the path of integration being the line $\mathrm{Im}\zeta = 0$.

From equation (3.6) we have

$$(3.8) \quad (\underline{N} - r\underline{D}) = \begin{bmatrix} \frac{1}{2}-r+\frac{1}{n}(r-1) & \frac{1}{2}+\frac{1}{n}(r-1) & & & & (\frac{1}{n}(r-1)) \\ \frac{1}{2}+\frac{1}{n}(r-1) & -r+\frac{1}{n}(r-1) & \frac{1}{2}+\frac{1}{n}(r-1) & & & \\ & \frac{1}{2}+\frac{1}{n}(r-1) & -r+\frac{1}{n}(r-1) & \frac{1}{2}+\frac{1}{n}(r-1) & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -r+\frac{1}{n}(r-1) & \frac{1}{2}+\frac{1}{n}(r-1) \\ (\frac{1}{n}(r-1)) & & & & \frac{1}{2}+\frac{1}{n}(r-1) & \frac{1}{2}-\frac{1}{n}(r-1) \end{bmatrix} .$$

Thus,

$$(3.9) \quad \underline{\Sigma}^{-1} - 2i\zeta\,(\underline{N} - r\underline{D}) = \begin{bmatrix} e+a & b+a & & & & (a) \\ b+a & c+a & b+a & & & \\ & b+a & c+a & b+a & & \\ & & \ddots & \ddots & \ddots & \\ & & & & c+a & b+a \\ (a) & & & & b+a & e+a \end{bmatrix}$$

$$= \underline{B}^{*}$$

where,

$$(3.10) \quad \begin{cases} e = 1 - i\zeta\,(1 - 2r) \\ a = \dfrac{2i}{n}\,\zeta\,(1 - r) \\ b = -\,(\rho + i\zeta) \\ c = 1 + \rho^2 + 2i\zeta r \quad . \end{cases}$$

Thus, by equation (3.7) , (ii)

$$(3.11) \qquad \varphi_u\,(\zeta) = \frac{(1 - \rho^2)^{\frac{1}{2}}}{|\underline{B}^*|^{\frac{1}{2}}} \quad .$$

To evaluate $|\underline{B}^*|$, we observe that $|\underline{B}^*|$ is of the form of the determinant R_n in Appendix II . By equation (II.14) , we have

$$(3.12) \qquad z + \frac{1}{z} = \frac{1 + \rho^2 + 2i\zeta r}{\rho + i\zeta} = -\frac{c}{b}$$

$$= \frac{1 + \rho^2 - 2\rho r + 2\rho r + 2i\zeta r}{\rho + i\zeta}$$

$$= \frac{1 + \rho^2 - 2\rho r}{\rho + i\zeta} + 2r \quad .$$

Thus,

$$(3.13) \qquad \rho + i\zeta = \frac{(1 + \rho^2 - 2\rho r)\,z}{(z^2 - 2zr + 1)} \quad .$$

From equations (II.10) , (II.11) and (II.15) , we get

$$(3.14) \qquad R_n = \left[\, 1 - a \left\{ \frac{2\,(e - b - c) - n(c + 2b)}{(c + 2b)^2} \right\} \,\right]$$

$$\times \; (eX_{n-1} - b^2 X_{n-2}) + \frac{a(e - c - b)^2}{(c + 2b)^2} \left\{ 2X_{n-1} + 2(-1)^{n-1} b^{n-1} \right\} ,$$

where,

$$X_n = \frac{(-b)^n}{(1 - z^2) z^n} \left[\, z^{2n} - z^2 - (z^{2n} - 1)\, z\, (-\tfrac{e}{b}) \,\right] .$$

We can now evaluate the determinant $|\underline{B}^*|$ by making the necessary substitutions into equation (3.14) from equation (3.10) .

Now, using equations (3.10) and (3.13) ,

$$(3.15) \qquad -\frac{e}{b} = \frac{1 - i\zeta\,(1 - 2r)}{(\rho + i\zeta)}$$

$$= \frac{1 + \rho(1 - 2r)}{\rho + i\zeta} + 2r - 1$$

$$= \frac{(1 + \rho - 2\rho r)(z^2 - 2zr + 1) + (2r - 1)(1 + \rho^2 - 2\rho r)\, z}{z\,(1 + \rho^2 - 2\rho r)}$$

$$= \frac{(\rho z - 1)(z - \rho + 2\rho r) + z^2 + 1 - 2\rho r z^2}{z\,(1 + \rho^2 - 2\rho r)} \quad ,$$

and equation (3.13) gives ,

$$(3.16) \qquad (-b)^n = (\rho + i\zeta)^n$$

$$= \frac{(1 + \rho^2 - 2\rho r)^n\, z^n}{(z^2 - 2zr + 1)^n} \quad .$$

Hence, equations (3.14) , (3.15) and (3.16) yield,

$$(3.17) \qquad X_n = \frac{(1 + \rho^2 - 2\rho r)^n}{(1 - z^2)(z^2 - 2zr + 1)^n} \Bigg[z^{2n} - z^2$$

$$- (z^{2n} - 1)\, \frac{\{(\rho z - 1)(z - \rho + 2\rho r) + z^2 + 1 - 2\rho r z^2\}}{(1 + \rho^2 - 2\rho r)} \Bigg]$$

$$= \frac{(1 + \rho^2 - 2\rho r)^{n-1}}{(1 - z^2)(z^2 - 2zr + 1)^n}$$

$$\times \Big[(z^{2n} - z^2)(1 + \rho^2 - 2\rho r) - (z^{2n} - 1)\{(\rho z - 1)(z - \rho + 2\rho r)$$

$$+ z^2 + 1 - 2\rho r z^2 \} \Big] \quad .$$

We also have the following required results

$$(3.18)\qquad e - c - b = 1 - i\zeta\,(1 - 2r) - (1 + \rho^2 + 2i\zeta r) + \rho + i\zeta$$

$$= \rho - \rho^2 \;,$$

$$(3.19)\qquad c + 2b = 1 + \rho^2 + 2i\zeta r - 2\,(\rho + i\zeta\,)$$

$$= 1 + \rho^2 - 2\rho r + 2(r - 1)(\rho + i\zeta)$$

$$= \frac{(1 + \rho^2 - 2\rho r)(z^2 - 2z + 1)}{(z^2 - 2zr + 1)} \;.$$

Equations (3.18) and (3.19) give

$$(3.20)\qquad \frac{e - c - b}{c + 2b} = \frac{(\rho - \rho^2)(z^2 - 2zr + 1)}{(1 + \rho^2 - 2\rho r)(z - 1)^2} \;.$$

Also,

$$(3.21)\qquad a = 2\,in^{-1}\,\zeta\,(1 - r)$$

$$= 2\,n^{-1}\,\{(r - 1)\,\rho - (r - 1)(\rho + i\zeta)\}$$

$$= \frac{2\,(r - 1)\,\{\rho\,(z^2 - 2zr + 1) - z(1 + \rho^2 - 2\rho r)\}}{n\,(z^2 - 2zr + 1)}$$

$$= \frac{2\,(r - 1)(\rho - z)(1 - \rho z)}{n\,(z^2 - 2zr + 1)} \;,$$

and, from equations (3.18) and (3.19) ,

$$(3.22)\qquad \frac{2\,(e - c - b) - n\,(c + 2b)}{(c + 2b)^2}$$

$$= \frac{(z^2 - 2zr + 1)^2\,[2(\rho - \rho^2) - n(1 + \rho^2 - 2\rho r)(z - 1)^2(z^2 - 2zr + 1)^{-1}]}{(1 + \rho^2 - 2\rho r)^2\,(1 - z)^4}$$

thus,

$$a \left\{ \frac{2(e - c - b) - n(c + 2b)}{(c + 2b)^2} \right\} \tag{3.23}$$

$$= \frac{2(r - 1)(\rho - z)(1 - \rho z)}{n(z^2 - 2zr + 1)} \cdot \frac{1}{(1 + \rho^2 - 2\rho r)^2 (1 - z)^4}$$

$$\times (z^2 - 2zr + 1)^2[2(\rho - \rho^2) - n(1 + \rho^2 - 2\rho r)(1 - z)^2(z^2 - 2zr + 1)^{-1}]$$

$$= \frac{2(r -1)(\rho - z)(1 - \rho z)}{n(1 + \rho^2 - 2\rho r)^2(1 - z)^4} \left\{ 2(\rho - \rho^2)(z^2 - 2zr + 1) \right.$$

$$\left. - n(1 + \rho^2 - 2\rho r)(1 - z)^2 \right\} \quad .$$

And,

$$e = 1 - i\zeta\ (1 - 2r) \tag{3.24}$$

$$= 1 + (1 - 2r)\ \rho - (1 - 2r)(\rho + i\zeta)$$

$$= \frac{(1 + \rho - 2\rho r)(z^2 - 2zr + 1) - (1 + \rho^2 - 2\rho r)\ z(1 - 2r)}{(z^2 - 2zr + 1)}$$

$$= \frac{(\rho z - 1)(z - \rho + 2\rho r) + z^2 + 1 - 2\rho r z^2}{(z^2 - 2zr + 1)} \quad .$$

After substituting equations (3.13) , (3.17) , (3.20) , (3.23) and (3.24) into equation (3.14) , and omitting terms relatively $O(n^{-1})$ we have

$$|\underline{B}^*| \sim \frac{(1 + \rho^2 - 2\rho r)^{n-3}}{(1 - z^2)(1 - z)^2(z^2 - 2zr + 1)^{n-1}} \tag{3.25}$$

$$\times \{(1 - \rho)(1 - \rho z)(1 + \rho + \rho z - z - 2\rho r)\}^2 \quad .$$

Therefore, substituting equation (3.25) into equation (3.11), we get

$$\varphi_u(\zeta(z)) \sim \frac{(1-\rho^2)^{\frac{1}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-3}{2}}} \times \frac{(1-z^2)^{\frac{1}{2}}(1-z)(z^2-2z\rho+1)^{\frac{n-1}{2}}}{(1-\rho z)(1-\rho)(1+\rho+\rho z-z-2\rho r)} \tag{3.26}$$

where,

$$\zeta(z) = i\rho - \frac{iz(1+\rho^2-2\rho r)}{(z^2-2zr+1)} \quad . \tag{3.27}$$

Now,

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{(1+\rho^2-2\rho r)(1-z^2)}{(z^2-2zr+1)(z-\rho)(1-\rho z)} \tag{3.28}$$

thus,

$$\frac{\varphi_u(\zeta(z))\,\zeta'(z)}{\zeta(z)} \sim \frac{(z^2-2zr+1)^{\frac{n-3}{2}}}{(1+\rho^2-2\rho r)^{\frac{n-5}{2}}} \times \frac{(1-z^2)^{3/2}(1-z)(1-\rho^2)^{\frac{1}{2}}}{(1-\rho z)^2(z-\rho)(1+\rho+\rho z-z-2\rho r)(1-\rho)} \tag{3.29}$$

As seen in Chapter II the transformation

$$z + \frac{1}{z} = \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta}$$

maps the ζ - plane cut from $-i\infty$ to $-i(1-\rho)^2/2(1-r)$ and from $i(1+\rho)^2/2(1+r)$ to $+i\infty$ onto the open disc $|z| < 1$, with the

cut being mapped onto the unit-circle $|z| = 1$.

Letting

$$K' = \frac{(1-\rho^2)^{\frac{1}{2}}}{(1-\rho)(1+\rho^2-2\rho r)^{\frac{n-5}{2}}} ,$$

$$g(z) = \frac{(1-z^2)^{3/2}(1-z)}{(1-\rho r)^2(1+\rho+\rho z-z-2\rho r)} ,$$

and

$$s = \frac{n-3}{2} ,$$

equation (3.29) is of the form

$$\text{(3.30)} \qquad \frac{\varphi_u(\zeta(z))\,\zeta'(z)}{\zeta(z)}\,dz \sim \frac{K'}{(z-\rho)}\,g(z)(1-2zr+z^2)^s\,dz$$

where, as before

1) $\zeta(z)$ represents the transformation

$$z + \frac{1}{z} = \frac{1+\rho^2+2ir\zeta}{\rho+i\zeta} ,$$

2) K' is a function of (ρ, r) ,

3) $g(z)$ is an algebraic function taking real values for real z and is regular in a simply-connected subset of $|z| \leq 1$, which contains $z = \rho$ and $z = r$,

4) s is a real variable taking large values.

Thus, by equations (2.24) and (2.25) the z-correspondent of the

ζ-integral in the inversion formula (3.7), (iii) is given by

$$(3.31) \qquad \frac{K'}{2\pi i} \int_{[\Gamma]} \frac{1}{(z-\rho)} g(z)(1-2rz+z^2)^s \, dz ,$$

where

$$\int_{[\Gamma]} = \lim_{\substack{a \xrightarrow{\Gamma} \rho \\ z^* \xrightarrow{\Gamma} r+i\sqrt{1-r^2}}} \left[\int_{\bar{z}^*}^{\bar{a}} + \int_a^{z^*} \right] ,$$

and

(i) $s = \dfrac{n-3}{2}$,

(ii) $K' = \dfrac{(1-\rho^2)^{\frac{1}{2}}}{(1-\rho)(1+\rho^2-2\rho r)^{\frac{n-5}{2}}}$,

(iii) $g(z) = \dfrac{(1-z^2)^{3/2}(1-z)}{(1-\rho z)^2(1+\rho+\rho z-z-2\rho r)}$.

Thus, by equations (2.35) and (2.36) the inversion formula (3.7), (iii) becomes, for $r < \rho$,

$$(3.32) \qquad H(r) = \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(\rho - z)} g(z)(1-2rz+z^2)^s \, dz ,$$

and, for $r > \rho$

$$(3.33) \qquad H(r) = 1 - \frac{K'}{2\pi i} \int_{\bar{A}\,[L]}^{A} \frac{1}{(z-\rho)} g(z)(1-2rz+z^2)^s \, dz ,$$

where the path of integration is, as before, the straight line from $\bar{A}$ to A.

With the integrals (3.32) and (3.33) satisfying the conditions of Chapter II, the general results previously obtained apply. Thus equations (2.69), (2.70) and (3.31) give in the present case, for $r \leq \rho$,

$$(3.34) \qquad H(r) \sim \frac{(1-\rho^2)^{-\frac{1}{2}}(1-r^2)^{n/2}}{2\sqrt{\pi}\,(1-\rho r)^2(1+\rho^2-2\rho r)^{\frac{n-5}{2}}}$$

$$\times \exp\left[\left(\frac{n-3}{2}\right)\frac{(\rho-r)^2}{1-r^2}\right]\int_{\left(\frac{n-3}{2}\right)\frac{(\rho-r)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp[-\nu]\,d\nu$$

and, for $r \geq \rho$

$$(3.35) \qquad H(r) \sim 1 - \frac{(1-\rho^2)^{-\frac{1}{2}}(1-r^2)^{n/2}}{2\sqrt{\pi}\,(1-\rho r)^2(1+\rho^2-2\rho r)^{\frac{n-5}{2}}}$$

$$\times \exp\left[\left(\frac{n-3}{2}\right)\frac{(r-\rho)^2}{1-r^2}\right]\int_{\left(\frac{n-3}{2}\right)\frac{(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}}\exp(-\nu]\,d\nu \;.$$

Now, by equations (2.77) and (2.78), with $\nu = \left(\frac{n-3}{2}\right)\frac{(\rho-r)^2}{1-r^2}$,

we have, for $r \leq \rho$,

$$(3.36) \qquad \exp\left[\left(\frac{n-3}{2}\right)\frac{(\rho - r)^2}{1 - r^2}\right] \int_{\left(\frac{n-3}{2}\right)\frac{(\rho-r)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu]\, d\nu$$

$$= \sqrt{2}\; \frac{\Phi\left[\dfrac{-\sqrt{n-3}\,(\rho - r)}{\sqrt{1-r^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-3}\,(\rho - r)}{\sqrt{1-r^2}}\right]}$$

and, for $r \geq \rho$

$$(3.37) \qquad \exp\left[\left(\frac{n-3}{2}\right)\frac{(r - \rho)^2}{1 - r^2}\right] \int_{\left(\frac{n-3}{2}\right)\frac{(r-\rho)^2}{1-r^2}}^{\infty} \nu^{-\frac{1}{2}} \exp[-\nu]\, d\nu$$

$$= \sqrt{2}\; \frac{\Phi\left[\dfrac{-\sqrt{n-3}\,(r - \rho)}{\sqrt{1-r^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-3}\,(r - \rho)}{\sqrt{1-r^2}}\right]} .$$

Thus, by equations (3.34) and (3.35) , the approximate distribution function of $\hat{\rho}$, for $r \leq \rho$, is given by,

$$(3.38) \qquad H(r) \sim \frac{1}{\sqrt{2\pi}} \frac{(1-\rho^2)^{-\frac{1}{2}}(1-r^2)^{n/2}}{(1-\rho r)^2(1+\rho^2-2\rho r)^{\frac{n-5}{2}}}$$

$$\times \frac{\Phi\left[\dfrac{-\sqrt{n-3}\,|\rho - r|}{\sqrt{1-r^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-3}\,|\rho - r|}{\sqrt{1-r^2}}\right]}$$

and, for $r \geq \rho$, by

$$(3.39) \qquad H(r) \sim 1 - \frac{1}{\sqrt{2\pi}} \; \frac{(1-\rho^2)^{-\frac{1}{2}}(1-r^2)^{n/2}}{(1-\rho r)^2(1+\rho^2-2\rho r)^{\frac{n-5}{2}}} \times \frac{\Phi\left[\dfrac{-\sqrt{n-3}\,|\rho - r|}{\sqrt{1-r^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-3}\,|\rho - r|}{\sqrt{1-r^2}}\right]} \; .$$

CHAPTER IV

THE APPROXIMATE POWER OF THE TEST FOR SERIAL INDEPENDENCE

For the linear Markov process given by (3.1) consider the hypothesis of serial independence, that is $\rho = 0$. To perform a test of this hypothesis we require a suitable testing procedure. That is a practical as well as dependable method for deciding to accept or to reject the hypothesis. As a criterion for selecting such a test, or testing procedure, NEYMAN and PEARSON (1933) introduced the concept of the power of a test. Simply, this is the probability of rejecting the hypothesis when it is false. Hence a desirable test is one with large power. In this chapter we investigate the approximate power of the test of serial independence based on the sample estimate of the serial correlation coefficient given by (3.4) and the approximate distribution function obtained for this estimate in Chapter III. The approximate power is obtained for selected values of the serial correlation coefficient ρ and the sample size n, from which the approximate power curves for the different values of n were sketched.

Consider the null hypothesis and the alternative hypothesis

$$(4.1) \qquad \begin{cases} H_o : \rho = 0 \\ H_1 : \rho \neq 0 \end{cases}$$

and the testing procedure

$$(4.2) \qquad \text{(i) Accept } H_1 \text{ if } |\hat{\rho}| = |\hat{\rho}(x_1, \ldots, x_n)| \geq c$$

$$\text{(ii) Accept } H_0 \text{ if } |\hat{\rho}| = |\hat{\rho}(x_1, \ldots, x_n)| < c$$

where the random variable $\hat{\rho}$ is the sample estimate of the serial correlation coefficient defined by equation (3.4) , and the real number c is determined such that

$$\Pr [|\hat{\rho}| \geq c \mid \rho = 0] = .05 \quad .$$

The power of the test for the alternative ρ_1 is given by

$$(4.3) \qquad P(\rho_1) = \Pr [|\hat{\rho}| \geq c \mid \rho = \rho_1] ,$$

and the power function of the test is the function of ρ , $P(\rho)$.

The approximate power function of the test for serial independence, in terms of the approximate distribution function of $\hat{\rho}$ obtained in Chapter III and given by equations (3.35) and (3.36) , is , for $-c \leq \rho \leq c$,

$$(4.4) \qquad P(\rho) = \frac{1}{\sqrt{2\pi}} \; \frac{(1-\rho^2)^{-\frac{1}{2}}(1-c^2)^{n/2}}{(1-\rho c)^2(1+\rho^2-2\rho c)^{\frac{n-5}{2}}}$$

$$\times \frac{\Phi\left[\dfrac{-\sqrt{n-3}\,|\rho-c|}{\sqrt{1-c^2}}\right]}{\varphi\left[\dfrac{-\sqrt{n-3}\,|\rho-c|}{\sqrt{1-c^2}}\right]} + \frac{1}{\sqrt{2\pi}}$$

$$\times \frac{(1-\rho^2)^{-\frac{1}{2}}(1-c^2)^{n/2}}{(1+\rho c)^2(1+\rho^2+2\rho c)^{\frac{n-5}{2}}}$$

$$\times \frac{\Phi\left[\frac{-\sqrt{n-3}\ |\rho + c|}{\sqrt{1-c^2}}\right]}{\varphi\left[\frac{-\sqrt{n-3}\ |\rho + c|}{\sqrt{1-c^2}}\right]}$$

and , for $-c \leq c \leq |\rho|$

(4.5) $$P(\rho) = 1 + \frac{1}{\sqrt{2\pi}} \frac{(1-|\rho|^2)^{-\frac{1}{2}}(1-c^2)^{n/2}}{(1-c|\rho|)^2(1+|\rho|^2+2c|\rho|)^{\frac{n-5}{2}}}$$

$$\times \frac{\Phi\left[\frac{-\sqrt{n-3}\ ||\rho| + c|}{\sqrt{1-c^2}}\right]}{\varphi\left[\frac{-\sqrt{n-3}\ ||\rho| + c|}{\sqrt{1-c^2}}\right]} - \frac{1}{\sqrt{2\pi}}$$

$$\times \frac{(1-|\rho|^2)^{-\frac{1}{2}}(1-c^2)^{n/2}}{(1-c|\rho|)^2(1+|\rho|^2-2c|\rho|)^{\frac{n-5}{2}}}$$

$$\times \frac{\Phi\left[\frac{-\sqrt{n-3}\ ||\rho| - c|}{\sqrt{1-c^2}}\right]}{\varphi\left[\frac{-\sqrt{n-3}\ ||\rho| - c|}{\sqrt{1-c^2}}\right]}$$

where c is such that

$$\frac{(1-c^2)^{n/2}}{\sqrt{2\pi}} \cdot \frac{\Phi\left[\frac{-\sqrt{n-3}\ |c|}{\sqrt{1-c^2}}\right]}{\varphi\left[\frac{-\sqrt{n-3}\ |c|}{\sqrt{1-c^2}}\right]} = 0.025 \quad .$$

Table 1. The Approximate Power of the Test for Serial Independence at a Significance Level of 5 % .

ρ \ n	10	15	20	25	35	50
0.0	0.050	0.050	0.050	0.050	0.050	0.050
0.1	0.057	0.063	0.069	0.075	0.087	0.150
0.2	0.082	0.106	0.132	0.157	0.210	0.288
0.3	0.129	0.191	0.253	0.316	0.434	0.567
0.4	0.212	0.334	0.451	0.530	0.645	0.790
0.5	0.351	0.515	0.589	0.672	0.811	0.927
0.6	0.502	0.587	0.703	0.780	0.918	0.981
0.7	0.498	0.660	0.800	0.890	0.970	0.996
0.8	0.473	0.722	0.870	0.943	0.900	0.999

In Table 1 we have tabulated the approximate power for the test (4.2) , using equations (4.4) and (4.5) , for selected values of the serial correlation coefficient ρ and sample size n . The approximate power function is symmetric about $\rho = 0$, and therefore we have calculated the approximate power only for positive values of ρ .

An examination of Table 1 discloses that, for a sample of size 10 , the approximate power decreases for values of ρ greater than 0.6 , instead of continuing to increase as would be expected. Although we have not been able to pinpoint the exact reason for this we suggest that a full investigation would reveal the approximate distribution function for $\hat{\rho}$, which has been obtained on the assumption of moderate ρ and r , and large n , suffers a considerable loss in accuracy here.

We also note that the approximate power curve for $n = 15$, illustrated in Appendix III, Fig. 1, has a distinct flattening of the tails, whereas the curves for the larger values of n do not exhibit this characteristic. This would indicate, that in the present context, a reasonable test could not be based on a sample of less than 20 observations.

In practise a test is usually made on a relatively small number of observations, generally 20 or less. This is due to the time and expense that would be otherwise involved. Now, Table 1 indicates that for a sample of 20 observations we would only be able to detect a ρ value of 0.5 about 60% of the time, whereas, for samples of 35 and 50 observations we could detect this value approximately 80% and 93 % of the time respectively. Even a ρ value of 0.8 would remain undetected 13% of the time while a sample of 50 observations would fail to detect this value in only 0.1% of the trails. Hence, we conclude, our testing procedure would only be of any real practical use in situations where large samples of observations could be conveniently applied.

BIBLIOGRAPHY

ABBASSI, S. A. (1964) "The Approximate Distribution of the Serial Correlation of a Pre-stationary Linear Markov Process", M.Sc. Thesis, University of Alberta (unpublished) .

ANDERSON, R. L. (1942) "Distribution of the Serial Correlation Coefficient", Ann. Math. Stat. 13, 1 - 13 .

DANIELS, H. E. (1956) "The Approximate Distribution of Serial Correlation Coefficients", Biometrika 43 , 169 - 185 .

ERDELYI, A., and WYMAN, M. (1963) "Method of Laplace", (To be published).

ERDELYI, A., and WYMAN, M. (1963a) "The Asymptotic Evaluation of Certain Integrals", Arch. Rat. Mech. Anal. 14, 217 - 260 .

GURLAND, J. (1948) "Inversion Formulae for the Distribution of Ratios", Ann. Math. Stat. 19, 228 - 237 .

NEYMAN, J., and PEARSON, E. S. (1933) "On the Problem of the Most Efficient Tests of Statistical Hypothesis", Trans. Roy. Soc. London, Series A, 231, 289 - 337 .

PATRICK, C. A. (1964) "Approximate Distribution Functions of Certain Sample Serial Correlation Coefficients", M.Sc. Thesis, University of Alberta (unpublished) .

PATTON, A. C. A.	(1961)	"The Approximate Distribution of the Ratios of the Square Successive Difference to the Square Difference in Observations from a Stationary Markov Process", M.Sc. Thesis, University of Alberta, (unpublished) .
SLATER, L. J.	(1960)	Confluent Hypergeometric Functions. Cambridge University Press .
SANSONE, G., and GERRETSEN, J.	(1960)	Lectures on the Theory of Functions of a Complex Variable. P. Nordhoff .
VON NEUMANN, J.	(1941)	"Distribution of the Ratio of the Mean Square Successive Difference to the Variance", Ann. Math. Stat. 12, 367 - 395 .
VON NEUMANN, J.	(1942)	"A Further Remark Concerning the Distribution of the Ratio of the Mean Square Successive Difference to the Variance", Ann. Math. Stat. 13, 86 - 88 .
WILLIAMS, J. D.	(1941)	"Moments of the Ratio of the Mean Square Difference in Samples from a Normal Universe", Ann. Math. Stat. 12, 239 - 241 .

APPENDIX I

CAUCHY'S PRINCIPAL VALUE OF AN INTEGRAL

Assumptions:

(1) The function $f(z)$ is defined in some region of the z-plane.

(2) For a fixed point $z = \rho$ and some fixed $R > 0$, $f(z)$ is regular for all points of the neighbourhood $|z - \rho| \leq R$ except at $z = \rho$.

(3) The point $z = \rho$ is a simple pole of $f(z)$.

(4) C is a continuous curve in the z-plane passing through $z = \rho$ and intersecting with $|z - \rho| = R$ at $z = A$ and $z = B$.

Consider the integral

$$\text{(I.1)} \qquad \lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_{\substack{A \\ [C]}}^{a} + \int_{\substack{b \\ [C]}}^{B} \right] f(z)\, dz$$

where the limits $a \to \rho$, $b \to \rho$ are taken on the set C.

We wish to determine sufficient conditons for which the above integral exists in the sense of a Cauchy Principal Value.

By assumption

$$f(z) = \frac{K}{z - \rho} + g(z)$$

for all z in $|z - \rho| \leq R$ where $g(z)$ is regular for all z in $|z - \rho| \leq R$, and K is constant with respect to z.

Thus,

$$\lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] f(z)\, dz$$

$$= \lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] \left(\frac{K}{z - \rho} + g(z) \right) dz$$

$$= \lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] \frac{K}{z - \rho}\, dz + \lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] g(z)\, dz \; .$$

Now, $g(z)$ is regular for all z in $|z - \rho| \leq R$ hence ,

$$\lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] g(z)\, dz = \int_A^B g(z)\, dz \quad .$$

Also,

$$\lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] \frac{K}{z - \rho}\, dz$$

$$= K \lim_{\substack{a \to \rho \\ b \to \rho}} \left[\log \left\{ \frac{a - \rho}{A - \rho} \cdot \frac{B - \rho}{b - \rho} \right\} \right]$$

$$= K \lim_{\substack{a \to \rho \\ b \to \rho}} \log \left\{ \frac{B - \rho}{A - \rho} \right\} + K \lim_{\substack{a \to \rho \\ b \to \rho}} \left\{ \frac{a - \rho}{b - \rho} \right\}$$

$$= K \log \left\{ \frac{B - \rho}{A - \rho} \right\} + K \lim_{\substack{a \to \rho \\ b \to \rho}} \left\{ \frac{a - \rho}{b - \rho} \right\} .$$

Thus,

$$\lim_{\substack{a \to \rho \\ b \to \rho}} \left[\int_A^a + \int_b^B \right] f(z)\, dz$$

$$= \lim_{\substack{a \to \rho \\ b \to \rho}} K \log \left\{ \frac{a - \rho}{b - \rho} \right\} + K \log \left\{ \frac{B - \rho}{A - \rho} \right\} + \int_A^B g(z)\, dz .$$

Hence, a sufficient condition that (I.1) exists as a Cauchy Principal Value is that

(I.2) $\displaystyle \lim_{\substack{a \to \rho \\ b \to \rho}} K \log \left\{ \frac{a - \rho}{b - \rho} \right\}$ is finite.

Consider the normal to the curve C at $z = \rho$, and the transformation

$$z - \rho = te^{i\theta}$$

where z, t and ρ are complex variables, and θ is a fixed angle determined so that the normal to the curve C at $z = \rho$ in the z-plane is transformed to be the line $\operatorname{Im} t = 0$ in the t-plane. Thus in the t-plane (I.2) becomes

$$\lim_{\substack{t_a \to 0 \\ t_b \to 0}} K \log \frac{t_a}{t_b}$$

where t_a , t_b are the t-images of a and b respectively, and the limits are through the set C_t (the t-image of C in the z-plane).

Now consider the special case where the curve C_t is symmetric about $\mathrm{Im}\, t = 0$. In this case $t_a = \bar{t}_b$ (the complex conjugate of t_b) .

Thus if $t_b = |t_b| e^{i\varphi}$

$$0 \leq \varphi \leq 2\pi$$

$$t_a = |t_b| e^{-i\varphi}$$

and

$\lim\limits_{\substack{t_a \to 0 \\ t_b \to 0}}$ becomes $\lim\limits_{\varphi \to \Phi}$.

Thus we have

$$\lim_{\substack{t_a \to 0 \\ t_b \to 0}} K \log \left\{ \frac{t_a}{t_b} \right\} = \lim_{\varphi \to \Phi} K \log \left\{ \frac{e^{-i\varphi}}{e^{i\varphi}} \right\}$$

$$= \lim_{\varphi \to \Phi} K \log (e^{-2i\varphi})$$

$$= \lim_{\varphi \to \Phi} - 2iK\varphi$$

$$= - 2iK\Phi$$

which is constant and finite .

[cf. SANSONE and GERRETSEN (1960), pp. 127 - 128]

APPENDIX II

EVALUATION OF A DETERMINANT

We wish to evaluate the determinant

$$
\text{(II.1)} \qquad R_n = \left|\begin{array}{cccccccc}
e+a & b+a & a & & & & & \\
b+a & c+a & b+a & & & (a) & & \\
a & b+a & c+a & & & & & \\
a & a & b+a & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & b+a & a & a \\
 & & & & & c+a & b+a & a \\
 & (a) & & & & b+a & c+a & b+a \\
 & & & & & a & b+a & e+a
\end{array}\right|_n .
$$

This may be written as the $(n + 1) \times (n + 1)$ determinant

$$
\text{(II.2)} \qquad R_{n+1} = \left|\begin{array}{cccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & e+a & b+a & \cdots & a & a & a \\
0 & b+a & c+a & \cdots & a & a & a \\
0 & a & b+a & \cdots & a & a & a \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & a & a & \cdots & b+a & a & a \\
0 & a & a & \cdots & c+a & b+a & a \\
0 & a & a & \cdots & b+a & c+a & b+a \\
0 & a & a & \cdots & a & b+a & e+a
\end{array}\right|_{n+1} .
$$

If we subtract "a" times the elements of the first row from each of the remaining rows we get

$$\text{(II.3)}\quad R_{n+1} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -a & e & b & \cdots & 0 & 0 & 0 \\ -a & b & c & \cdots & 0 & 0 & 0 \\ -a & 0 & b & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a & 0 & 0 & \cdots & b & 0 & 0 \\ -a & 0 & 0 & \cdots & c & b & 0 \\ -a & 0 & 0 & \cdots & b & c & b \\ -a & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -a & e & b & \cdots & 0 & 0 & 0 \\ -a & b & c & \cdots & 0 & 0 & 0 \\ -a & 0 & b & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a & 0 & 0 & \cdots & b & 0 & 0 \\ -a & 0 & 0 & \cdots & c & b & 0 \\ -a & 0 & 0 & \cdots & b & c & b \\ -a & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_{n+1}$$

$$+ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & e & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ 0 & 0 & b & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 0 & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_{n+1} ,$$

or

(II.4) $$R_n = - a\, P_{n+1} + Q_n \quad ,$$

where

$$P_{n+1} = \begin{array}{c|c} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & e & b & \cdots & 0 & 0 & 0 \\ 1 & b & c & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdots & c & b & 0 \\ 1 & 0 & 0 & \cdots & b & c & b \\ 1 & 0 & 0 & \cdots & 0 & b & e \end{vmatrix} & n + 1 \end{array}$$

and

$$Q_n = \begin{array}{c|c} \begin{vmatrix} e & b & 0 & \cdots & 0 & 0 & 0 \\ b & c & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 0 & 0 & 0 & \cdots & 0 & b & e \end{vmatrix} & n \end{array} .$$

Multiplying the sum of rows 2 to $(n + 1)$ of P_{n+1} by $(-1)/(c + 2b)$ and adding to the first row, we have

$$P_{n+1} = \begin{vmatrix} \frac{-n}{c+2b} & 1 - \frac{e+b}{c+2b} & 0 & \cdots & 0 & 0 & 1 - \frac{e+b}{c+2b} \\ 1 & e & b & \cdots & 0 & 0 & 0 \\ 1 & b & c & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & c & b & 0 \\ 1 & 0 & 0 & \cdots & b & c & b \\ 1 & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_{n+1} .$$

Multiplying the sum of columns 2 to $(n + 1)$ of P_{n+1} by $(-1)/(c + ab)$ and adding the result to the first column, we have

$$P_{n+1} = \begin{vmatrix} \frac{2(e-b-c) - n(c+2b)}{(c+2b)^2} & 1 - \frac{e+b}{c+2b} & 0 & \cdots & 0 & 0 & 1 - \frac{e+b}{c+2b} \\ 1 - \frac{e+b}{c+2b} & e & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 1 - \frac{e+b}{c+2b} & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_{n+1} ,$$

or

$$\text{(II.5)} \qquad P_{n+1} = \frac{2(e - b - c) - n(c + 2b)}{(c + 2b)^2} Q_n + \frac{e - b - c}{(c + 2b)} M_n + (-1)^{n-1} \frac{e - b - c}{(c + 2b)} N_n ,$$

where,

$$(II.6) \quad M_n = \begin{vmatrix} \frac{c+b-e}{c+2b} & 0 & 0 & \cdots & 0 & 0 & \frac{c+b-e}{c+2b} \\ b & c & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 0 & 0 & 0 & \cdots & 0 & b & e \end{vmatrix}_n$$

and

$$(II.7) \quad N_n = \begin{vmatrix} \frac{c+b-e}{c+2b} & 0 & 0 & \cdots & 0 & 0 & \frac{c+b-e}{c+2b} \\ e & b & 0 & \cdots & 0 & 0 & 0 \\ b & c & b & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \end{vmatrix}_n .$$

Expanding in terms of the elements of the first row, we get

$$(II.8) \qquad M_n = \frac{c + b - e}{c + 2b} X_{n-1} + (-1)^{n-1} \frac{c + b - e}{c + 2b} b^{n-1}$$

and

$$(II.9) \qquad N_n = \frac{c + b - e}{c + 2b} b^{n-1} + (-1)^{n-1} \frac{c + b - e}{c + 2b} X_{n-1} ,$$

where,

$$X_{n-1} = \begin{vmatrix} e & b & 0 & \cdots & 0 & 0 & 0 \\ b & c & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 0 & 0 & 0 & \cdots & 0 & b & c \end{vmatrix}_{n-1} .$$

Hence

$$\text{(II.10)} \qquad R_n = \left[1 - a \left\{ \frac{2(e - b - c) - n(c + 2b)}{(c + 2b)^2} \right\} \right] Q_n + \frac{a(e - c - b)^2}{(c + 2b)^2} \left\{ 2X_{n-1} + 2(-1)^{n-1} b^{n-1} \right\} ,$$

where,

$$\text{(II.11)} \qquad Q_n = e\, X_{n-1} - b^2 X_{n-2} .$$

Now, from equation (II.9) ,

$$\text{(II.12)} \quad X_n = \begin{vmatrix} e & b & 0 & \cdots & 0 & 0 & 0 \\ b & c & b & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & \cdots & b & c & b \\ 0 & 0 & 0 & \cdots & 0 & b & c \end{vmatrix}_{n}$$

$$= (-b)^n \begin{vmatrix} -\frac{e}{b} & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -\frac{c}{b} & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -\frac{c}{b} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{c}{b} & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & -\frac{c}{b} & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -\frac{c}{b} \end{vmatrix}_n$$

$$= (-b)^n A_n \quad .$$

Thus, if A_j is the determinant consisting of the first j rows and j columns of A_n $(j \leq n)$, we have

(i) $\quad A_1 = -\frac{e}{b}$,

(ii) $\quad A_2 = (\frac{c}{b})(\frac{e}{b}) - 1$,

and in general

(iii) $\quad A_j = (-\frac{c}{b}) A_{j-1} - A_{j-2}$.

The solution of this difference equation is

$$A_j = \left[\frac{1 + (\frac{e}{b}) z}{1 - z^2} \right] z^n + \left[\frac{(-\frac{e}{b} - z) z}{1 - z^2} \right] z^{-n} \tag{II.13}$$

where,

$$z + \frac{1}{z} = -\frac{c}{b} \quad . \tag{II.14}$$

Hence, equation (II.12) gives

$$\text{(II.15)} \qquad X_n = (-b)^n \left[\left\{ \frac{1 + (\frac{e}{b})\, z}{1 - z^2} \right\} z^n + \left\{ \frac{(-\frac{e}{b} - z)\, z}{1 - z^2} \right\} z^{-n} \right]$$

$$= \frac{(-b)^n}{(1 - z^2)\, z^n} \left[z^{2n} - z^2 - (z^{2n} - 1)\, z\, (-\tfrac{e}{b}) \right] .$$

Thus, from equations (II.10) , (II.11) and (II.15) , we get

$$\text{(II.16)} \qquad R_n = \left[1 - a \left\{ \frac{2\,(e - b - c) - n\,(c + 2b)}{(c + 2b)^2} \right\} \right]$$

$$\times \left[\frac{e\,(-b)^{n-1}}{(1 - z^2)\, z^{n-1}} \left\{ z^{2n-2} - z^2 - (z^{2n-2} - 1)\, z\, (-\tfrac{e}{b}) \right\} \right.$$

$$\left. + \frac{(-1)^{n-1}\, b^n}{(1 - z^2)\, z^{n-2}} \left\{ z^{2n-4} - z^2 - (z^{2n-4} - 1)\, z\, (-\tfrac{e}{b}) \right\} \right]$$

$$+ \frac{2a\,(e - c - b)^2}{(c + 2b)^2} \left[\frac{(-b)^{n-1}}{(1 - z^2) z^{n-1}} \left\{ z^{2n-2} - z^2 - (z^{2n-2} - 1) \right. \right.$$

$$\left. \left. \times\, z\, (-\tfrac{e}{b}) \right\} + (-1)^{n-1}\, b^{n-1} \right] .$$

APPENDIX III

POWER CURVES

This appendix serves to illustrate the data of Table 1 . We reproduce the power curve for the cases, $n = 15, 20, 25, 35$ and 50 , omitting the case of $n = 10$ as our results are not valid here. The curves are given for $-0.8 \leq \rho \leq 0.8$, and are symmetric about the power axis, that is, the alternative ρ has the same power as the alternative $-\rho$, for each ρ . The curves give us an overall indication of the increase in power as the serial correlation coefficient ρ and sample size n become larger. Giving us an idea of the effectiveness of the test for various n .

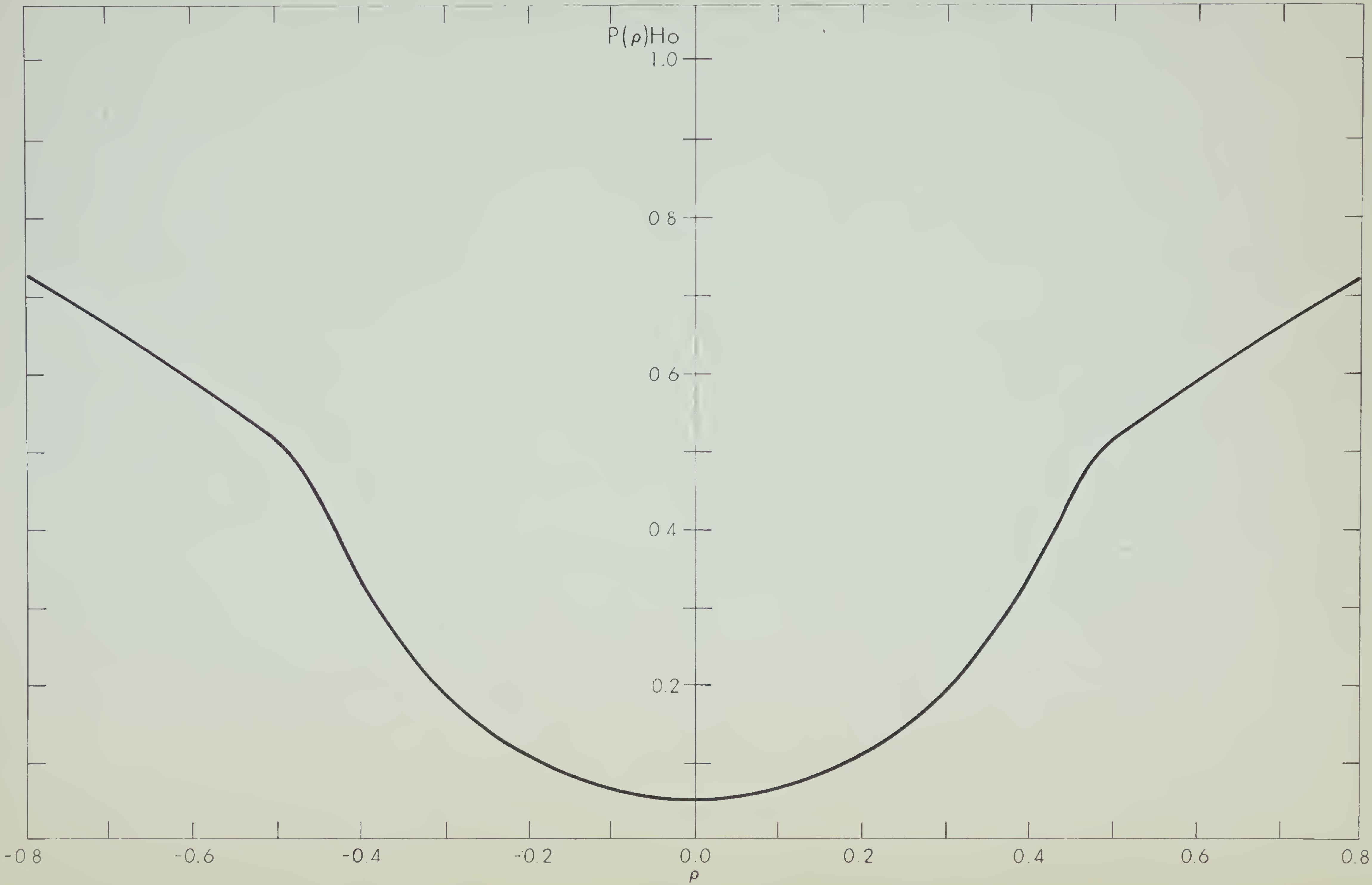


Figure 1. Power Function for $n = 15$

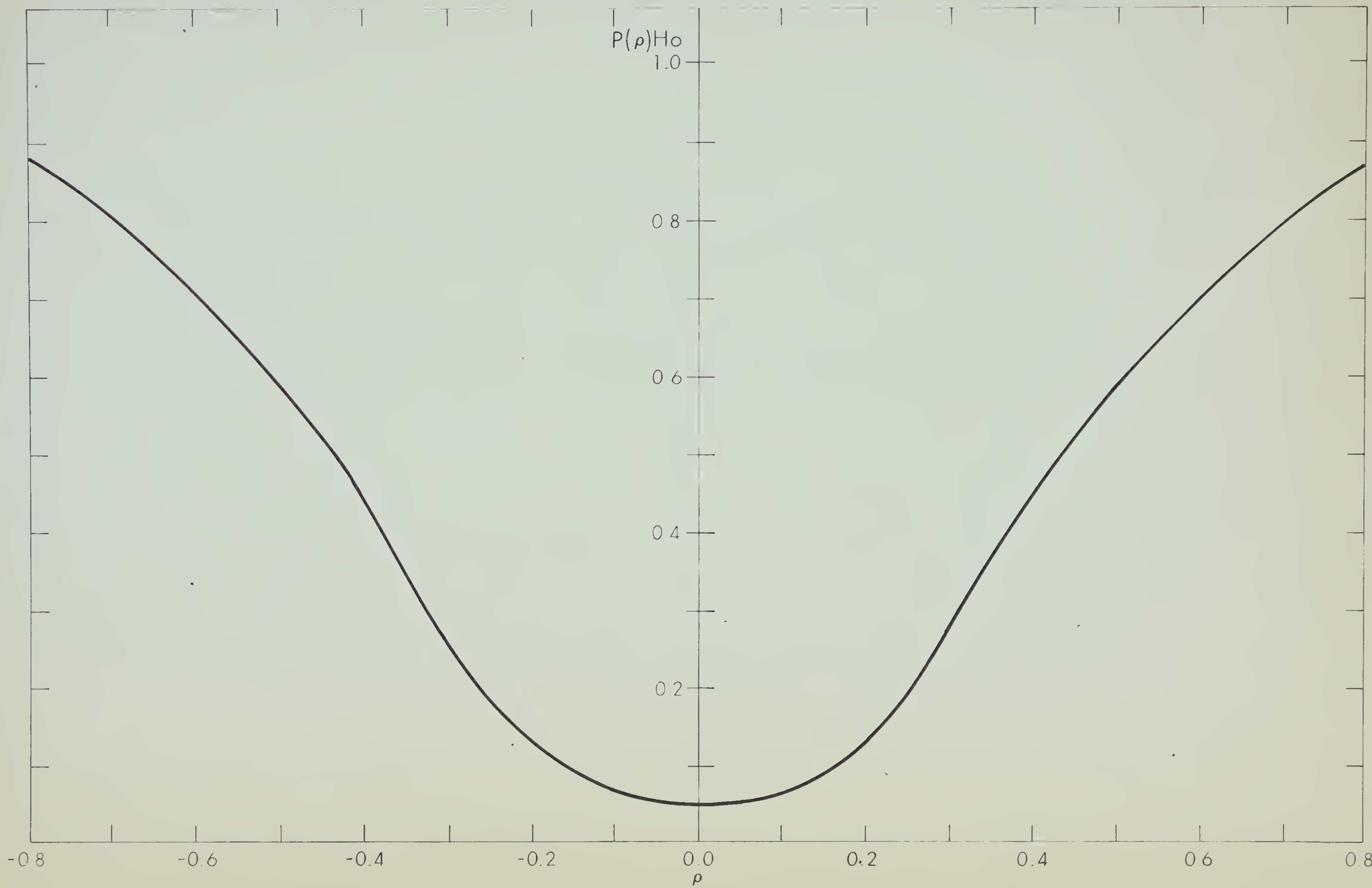


Figure 2 Power Function for n = 20

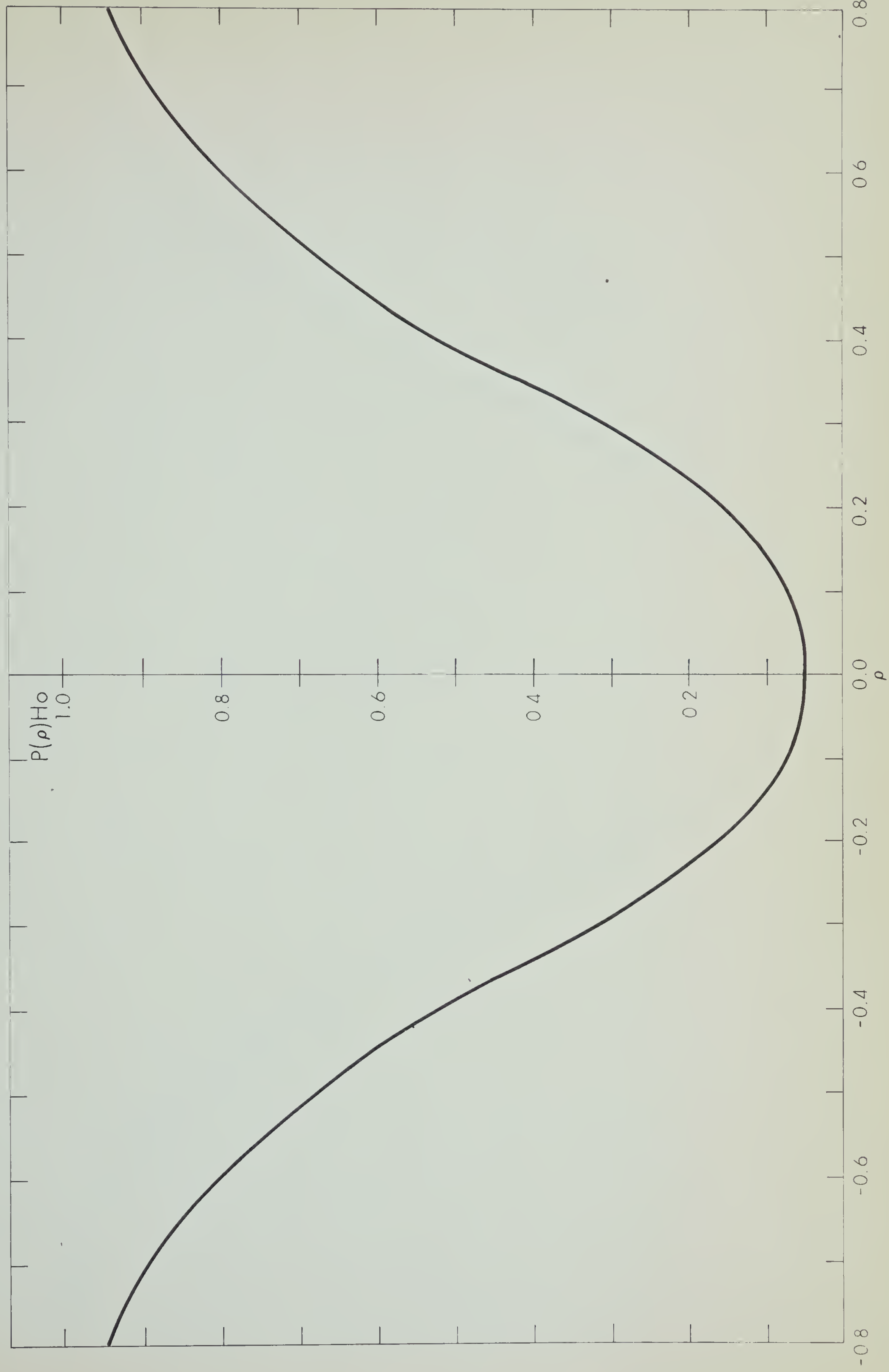


Figure 3. Power Function for n = 25

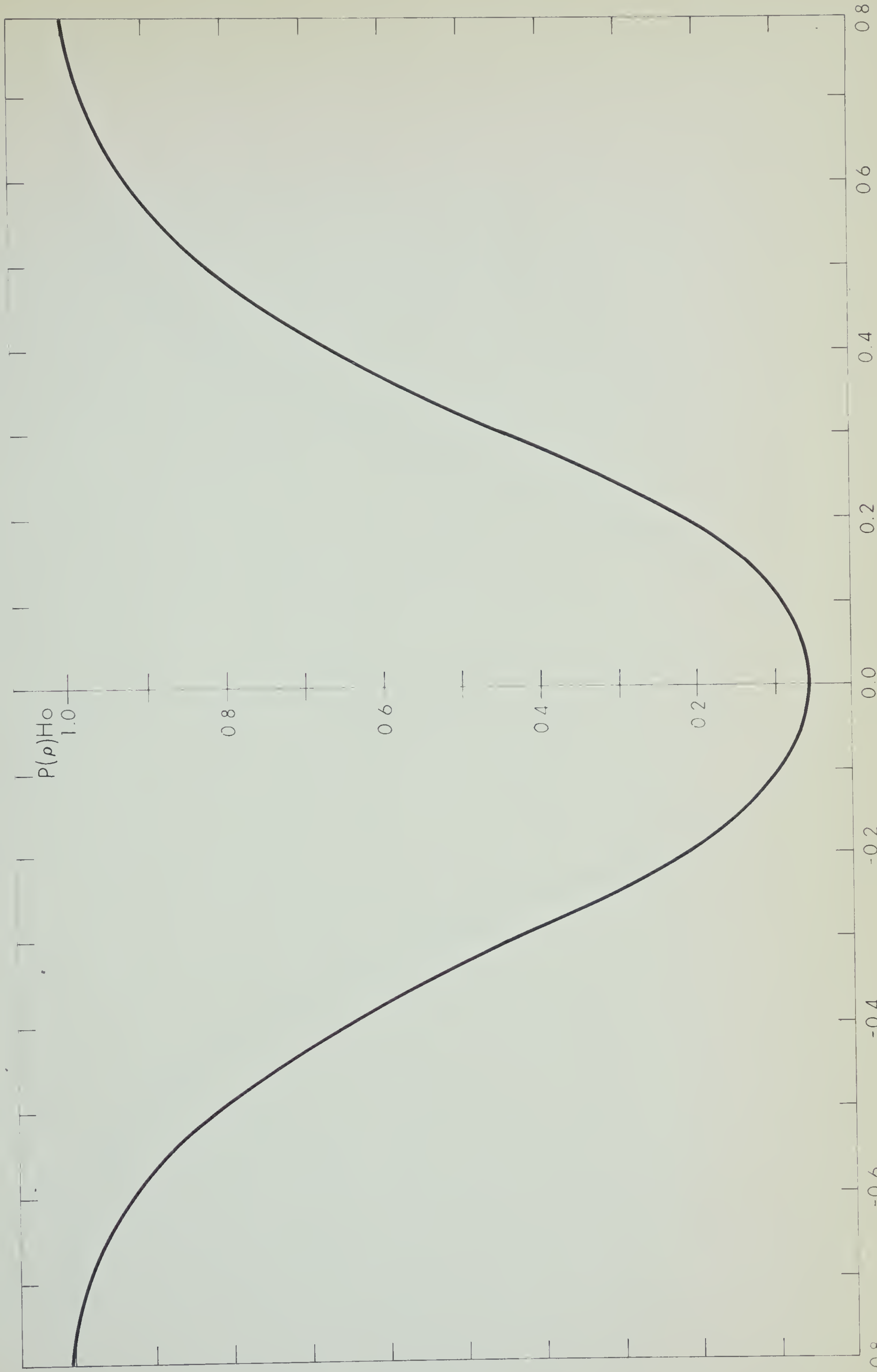


Figure 4 Power Function for n = 35

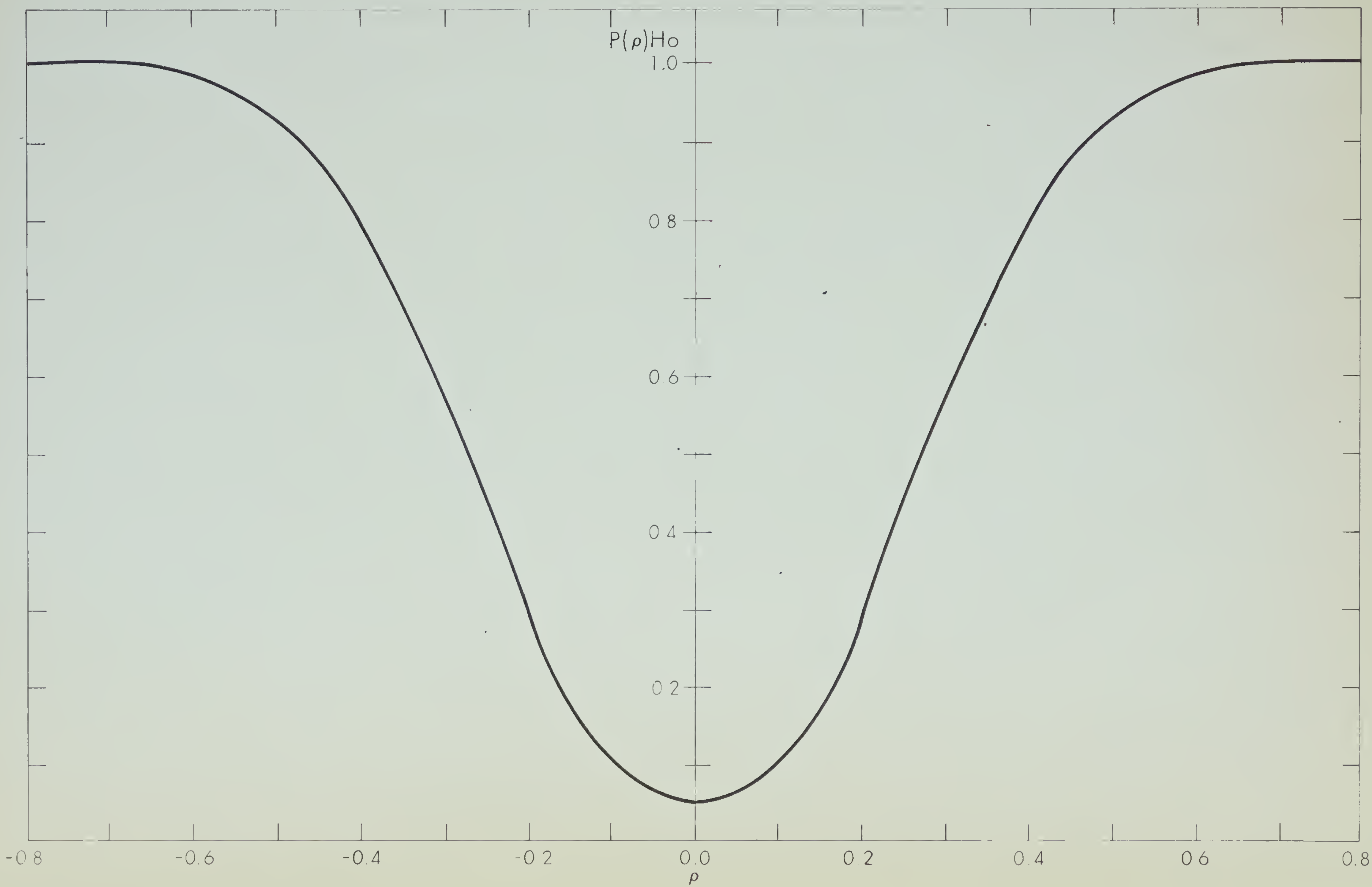


Figure 5 Power Function for $n = 50$